

THE 3-BALL IS A LOCAL PESSIMUM FOR PACKING

YOAV KALLUS

ABSTRACT. It was conjectured by Ulam that the ball has the lowest optimal packing fraction out of all convex, three-dimensional solids. Here we prove that any origin-symmetric convex solid of sufficiently small asphericity can be packed at a higher efficiency than balls. We also show that in dimensions 4, 5, 6, 7, 8, and 24 there are origin-symmetric convex bodies of arbitrarily small asphericity that cannot be packed using a lattice as efficiently as balls can be.

1. INTRODUCTION

When Martin Gardner’s *New Mathematical Diversions*, collecting some of his *Mathematical Games* columns, was reprinted in 1995, the column on “Packing Spheres” appeared with a postscript, in which Gardner writes “Stanislaw Ulam told me in 1972 that he suspected the sphere was the worst case of dense packing of identical convex solids, but that this would be difficult to prove” [4]. In other words, Ulam “suspected that spheres, in their densest packing, allow more empty space than the densest packing of any identical convex solids” [5]. For the purposes of this article, we restrict our attention only to solids that are origin-symmetric and only to lattice packings. Since too many conjectures already bear the name *Ulam’s conjecture*, it would be appropriate to refer to this one as *Ulam’s packing conjecture*, or perhaps even as *Ulam’s last conjecture*, seeing how it too was published posthumously and how the space in Gardner’s postscript evidently could not fit any motivation for Ulam’s suspicion.

The suspicion is not hard to motivate naively by the fact that the sphere is the most symmetric solid and therefore also the least free: in placing a sphere in space there are only three degrees of freedom, compared to five in the case of any other solid of revolution, and six

Date: July 22, 2014.

Key words and phrases. Convex body, Spherical harmonics, Packing, Sphere packing, Lattice, Pessimism.

Final submitted manuscript, to be published in *Advances in Mathematics*, <http://dx.doi.org/10.1016/j.aim.2014.07.015>.

in the case of any other solid. Therefore, it is natural to suspect that if we break the rotational symmetry of the sphere, we introduce more freedom, which could be used to tighten up the packing and bring the packing fraction above the optimal packing fraction of spheres.

That in the plane the circle can be packed more densely than some other domains belies this naive motivation, which one would expect to work equally well in the plane. A packing of circles can cover $0.9068\dots$ of the plane, while a packing of regular octagons cannot cover more than $0.9061\dots$. In 1934, Reinhardt conjectured that a smoothed octagon he constructed has the lowest optimal packing fraction of any origin-symmetric convex domain [11]. A packing of smoothed octagons can cover at most $0.9024\dots$ of the plane. In fact, the circle is not even a *local* pessimum (the opposite of an optimum). There are origin-symmetric shapes arbitrarily circular (the outradius and inradius both being arbitrarily close to 1) that cannot be packed as efficiently as circles [8]. Ulam's conjecture implies that this would not be the case in three dimensions. The question then is in which dimensions, if in any, the ball is a local pessimum. In this article, we answer this question for all dimensions in which the highest lattice packing fraction for spheres is known.

In Section 2 we provide some preliminaries about convex bodies and admissible lattices. In Section 3 we introduce the notions of perfection and eutaxy and their implications for the question at hand. We observe that these implications are strong enough to determine that the ball is not a local pessimum in dimensions 4, 5, 6, 7, 8, and 24. The 3-dimensional case is treated in Section 4, where the ball is proved to be a local pessimum.

2. CONVEX BODIES AND ADMISSIBLE LATTICES

An n -dimensional *convex body* is a convex, compact subset of \mathbb{R}^n with a nonempty interior. A planar body is a *domain*, and a 3-dimensional body is a *solid*. A body K is *symmetric about the origin* (or origin-symmetric) if $-K = K$. In this article we discuss only such bodies, and we will implicitly assume that every body mentioned is symmetric about the origin. We denote by B^n the Euclidean unit ball of \mathbb{R}^n . The space of origin-symmetric convex bodies \mathcal{K}_0^n in \mathbb{R}^n is a metric space equipped with the Hausdorff metric $\delta_H(K, K') = \min\{\varepsilon : K \subseteq K' + \varepsilon B^n, K' \subseteq K + \varepsilon B^n\}$. The set of bodies K satisfying $aB^n \subseteq K \subseteq bB^n$ for $b > a > 0$ is compact [6].

Let $S^{n-1} = \partial B^n$ be the unit sphere. The value of the *support function* of an n -dimensional body in the direction $\mathbf{x} \in S^{n-1}$ is given by $h_K(\mathbf{x}) =$

$\max_{\mathbf{y} \in K} \langle \mathbf{x}, \mathbf{y} \rangle$. The half-space $H_K(\mathbf{x}) = \{\mathbf{y} : \langle \mathbf{x}, \mathbf{y} \rangle \leq h_K(\mathbf{x})\}$ contains K , and a body is uniquely determined by its support function since $K = \bigcap_{\mathbf{x} \in S^{n-1}} H_K(\mathbf{x})$. Similarly, the value of *radial function* of a body in the direction $\mathbf{x} \in S^{n-1}$ is given by $r_K(\mathbf{x}) = \max_{\lambda \mathbf{x} \in K} \lambda$. Again, a body is uniquely determined by its radial function. For origin-symmetric bodies, both the support function and the radial function are even functions.

An n -dimensional *lattice* is the image of the integer lattice \mathbb{Z}^n under some non-singular linear map T . The determinant $d(\Lambda)$ of a lattice $\Lambda = T\mathbb{Z}^n$ is the volume of the image of the unit cube under T and is given by $d(\Lambda) = |\det T|$. The space \mathcal{L}^n of n -dimensional lattices can be equipped with the metric $\delta(\Lambda, \Lambda') = \min\{\|T - T'\| : \Lambda = T\mathbb{Z}^n, \Lambda' = T'\mathbb{Z}^n\}$, where $\|\cdot\|$ is the Hilbert-Schmidt norm. A lattice is called *admissible* for a body K if it intersects the interior of K only at the origin. A lattice Λ is admissible for K if and only if $\{K + 2\mathbf{l} : \mathbf{l} \in \Lambda\}$ is a packing, i.e., $K + 2\mathbf{l}$ and $K + 2\mathbf{l}'$ have disjoint interiors for $\mathbf{l}, \mathbf{l}' \in \Lambda$, $\mathbf{l} \neq \mathbf{l}'$. The fraction of space covered by this packing is $2^{-n} \text{vol } K/d(\Lambda)$. The set of lattices Λ , such that Λ is admissible for a body K and $d(\Lambda) < a$ for some $a > 0$, is a compact set [6].

The *critical determinant* d_K is the minimum, necessarily attained due to compactness, of all determinants of lattices admissible for K . A lattice attaining this minimum is called a *critical lattice* of K . Clearly, if $K' \subseteq K$, then $d_{K'} \leq d_K$. If this inequality is strict whenever K' is a proper subset of K , we say that K is an *irreducible* body. The optimal packing fraction for K is $\varphi(K) = 2^{-n} \text{vol } K/d_K$. Note that $\varphi(TK) = \varphi(K)$ for any nonsingular linear transformation T . Therefore, we may define φ as a function over the space of linear classes of n -dimensional bodies, equipped with the Banach-Mazur distance $\delta_{BM}([K], [L]) = \min\{t : L' \subseteq K' \subseteq e^t L', K' \in [K], L' \in [L]\}$. Since this space is compact, there must be a body K with the lowest possible optimal packing fraction among all n -dimensional bodies. Such a body is called a *global pessimum* for packing. If a body attains the lowest possible optimal packing fraction among bodies in a neighborhood of itself with respect to Hausdorff distance, then we say it is a *local pessimum* for packing. A locally pessimal body is necessarily irreducible, but the converse is not necessarily true.

In two dimensions, Reinhardt's smoothed octagon is known to be locally pessimal and is conjectured to be globally pessimal [10]. The unit disk in two dimensions is irreducible but not locally pessimal [8]. Below we show that the unit ball is locally pessimal for $n = 3$, irreducible but

not locally pessimal for $n = 4$ and 5 , and reducible for $n = 6, 7, 8$, and 24 .

3. PERFECTION AND EUTAXY

Let $S \subset S^{n-1}$ be a finite, origin-symmetric set of unit vectors (that is $\mathbf{x} \in S$ if and only if $-\mathbf{x} \in S$). Every vector of S defines a projection map $P_{\mathbf{x}}(\cdot) = \langle \cdot, \mathbf{x} \rangle \mathbf{x}$. If these projection maps span the space Sym^n of symmetric linear maps $\mathbb{R}^n \rightarrow \mathbb{R}^n$, we say that S is a *perfect* configuration. The space Sym^n , equipped with inner product $\langle Q, Q' \rangle = \text{trace}(QQ')$ is isomorphic to $\mathbb{R}^{n(n+1)/2}$. Therefore, a perfect configuration must have at least $n(n+1)$ vectors [9].

If there are numbers $v_{\mathbf{x}}, \mathbf{x} \in S$, such that

$$(1) \quad v_{\mathbf{x}} = v_{-\mathbf{x}} \text{ and } \text{Id} = \sum_{\mathbf{x} \in S} v_{\mathbf{x}} P_{\mathbf{x}},$$

then we say that S is a *weakly eutactic* configuration and we call the coefficients $v_{\mathbf{x}}$ eutaxy coefficients [9]. A configuration is *eutactic* (resp. *semi-eutactic*) if it has positive (resp. non-negative) eutaxy coefficients. For reasons that would become apparent with Corollary 1, a configuration that is perfect and eutactic is called *extreme*. A configuration S is semi-eutactic if and only if the identity map is in the closed cone generated by $\{P_{\mathbf{x}} : \mathbf{x} \in S\}$, and extreme if and only if the identity is in the interior of this cone.

Suppose that S is extreme, then we say that S is redundantly semi-eutactic (resp. redundantly extreme) if the configuration $S \setminus \{\mathbf{x}, -\mathbf{x}\}$ is semi-eutactic (resp. extreme) for all $\mathbf{x} \in S$. On the other hand, if S is extreme and has the minimal number of vectors $n(n+1)$, we say the configuration is *minimally extreme*. The eutaxy coefficients of a minimally extreme configuration are unique and the configuration is not redundantly semi-eutactic.

The *minimal norm* $m(\Lambda)$ of a lattice Λ is the minimum of $\langle \mathbf{x}, \mathbf{x} \rangle$ over $\mathbf{x} \in \Lambda \setminus \{0\}$, and the set of points $S(\Lambda) \subseteq \Lambda$ that achieve this value are the *minimal vectors* of Λ . We say of a lattice Λ that it is perfect (resp. eutactic, semi-eutactic, and so on), if the configuration of its minimal vectors, scaled to lie on the unit sphere, is perfect (resp. eutactic, semi-eutactic, and so on).

In dimensions $n \leq 8$ and $n = 24$, the critical lattices of B^n are known. Admissibility for B^n is invariant under rotations, and so the critical lattice can be unique only up to rotations, which in fact it is in these dimensions. For $n = 2, 3, 4, 5, 6, 7, 8$, the critical lattices of B^n are given by the root lattices $A_2, D_3, D_4, D_5, E_6, E_7$, and E_8 [2]. The

root lattices are most easily specified (up to rotation) in terms of their Dynkin diagrams:

$$\begin{aligned}
 A_n & \quad \circ - \circ - \cdots - \circ - \circ & \text{for } n = 2, 3, \dots, \\
 & \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \quad \quad \mathbf{a}_{n-1} \quad \mathbf{a}_n \\
 (2) \quad D_n & \quad \circ - \circ - \cdots - \circ - \circ & \text{for } n = 3, 4, \dots, \\
 & \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \quad \quad \mathbf{a}_{n-2} \quad \mathbf{a}_{n-1} \\
 & \quad \quad \quad \quad \quad \quad \quad \quad \circ \\
 & \quad \quad \quad \quad \quad \quad \quad \quad | \quad \mathbf{a}_n \\
 E_n & \quad \circ - \cdots - \circ - \circ - \circ & \text{for } n = 6, 7, 8. \\
 & \quad \mathbf{a}_1 \quad \quad \quad \mathbf{a}_{n-3} \quad \mathbf{a}_{n-2} \quad \mathbf{a}_{n-1} \\
 & \quad \quad \quad \quad \quad \quad \quad \quad \circ \\
 & \quad \quad \quad \quad \quad \quad \quad \quad | \quad \mathbf{a}_n
 \end{aligned}$$

Here the nodes correspond to generating vectors of the lattice, i.e., $\Lambda = \{\sum_{i=1}^n z_i \mathbf{a}_i : z_i \in \mathbb{Z}\}$, and their inner products are determined by the edges: $\langle \mathbf{a}_i, \mathbf{a}_j \rangle = 1, -\frac{1}{2}, 0$ respectively when $i = j$, \mathbf{a}_i is connected by an edge to \mathbf{a}_j , and otherwise. Note that A_3 and D_3 are identical. The lattice A_n can be embedded isometrically as an n -dimensional sublattice of the $(n+1)$ -dimensional integer lattice \mathbb{Z}^{n+1} , namely

$$(3) \quad A_n \simeq \{2^{-1/2} \mathbf{z} : \mathbf{z} \in \mathbb{Z}^{n+1}, \sum_{i=1}^{n+1} z_i = 0\}.$$

A particular rotation of the lattice D_n is given by

$$(4) \quad D_n = \{2^{-1/2} \mathbf{z} : \mathbf{z} \in \mathbb{Z}^n, \sum_{i=1}^n z_i \in 2\mathbb{Z}\}.$$

In 24 dimensions, the Leech lattice Λ_{24} is the unique critical lattice of B^n [1]. The Leech lattice can be constructed as a union of two translates, $\Lambda_{24} = L \cup (L + \mathbf{x})$, where

$$\begin{aligned}
 (5) \quad L & = \{2^{-3/2} \mathbf{z} : \mathbf{z} \in \mathbb{Z}^{24}, \sum_{i=0}^{24} z_i \in 4\mathbb{Z}, \mathbf{z} - \mathbf{c} \in (2\mathbb{Z})^n, \mathbf{c} \in \mathcal{C}\} \\
 \mathbf{x} & = 2^{-5/2}(-3, 1, 1, \dots, 1),
 \end{aligned}$$

and $\mathcal{C} \subseteq \{0, 1\}^{24}$ is the extended binary Golay code of size 2^{12} and minimum weight 8 [2]. All the root lattices and the Leech lattice are extreme [2]. In fact, since the automorphism group $\text{Aut}(\Lambda)$ acts transitively on the minimal vectors $S(\Lambda)$ in all these cases, it is possible to pick all the eutaxy coefficients equal to $n/|S(\Lambda)|$ [2]. We now establish which of these lattices are minimally extreme, redundantly extreme, and redundantly semi-eutactic.

Lemma 1. *The lattices A_n are minimally extreme for all $n = 2, 3, \dots$. The lattices D_n are redundantly semi-eutactic and not redundantly extreme for all $n = 4, 5, \dots$. The lattices E_6, E_7, E_8 , and Λ_{24} are redundantly extreme.*

Proof. The minimal vectors of the embedding of A_n in \mathbb{R}^{n+1} given in (3) are all the vectors derived from $2^{-1/2}(1, -1, 0, 0, \dots, 0)$ by coordinate permutations. There are exactly $n(n+1)$ such vectors and therefore A_n is minimally extreme.

The minimal vectors of D_n , as embedded in \mathbb{R}^n by (4), are all $2n(n-1)$ permutations of vectors of the form $2^{-1/2}(\pm 1, \pm 1, 0, 0, \dots, 0)$. We want to show that for all $\mathbf{x} \in S(D_n)$, the configuration $S(D_n) \setminus \{\pm \mathbf{x}\}$ is semi-eutactic but not extreme. Note that because $\text{Aut}(D_n)$ acts transitively on $S(D_n)$, $S(D_n) \setminus \{\pm \mathbf{x}\}$ is semi-eutactic (extreme) if and only if $S(D_n) \setminus \{\pm \mathbf{x}'\}$ is. Consider the linear system (1) for the eutaxy coefficients of $S(D_n)$. When $n \geq 4$, it has more variables than equations and has at least one positive solution, $\mathbf{u}_0 = (\frac{1}{2(n-1)})_{\mathbf{x} \in S(D_n)}$. Therefore, the solution space must contain a line $\mathbf{u}_0 + t\mathbf{u}_1$, which in turn must contain a non-negative solution with at least one pair of vanishing coefficients $v_{\pm \mathbf{x}_0} = 0$. Therefore, $S(D_n) \setminus \{\pm \mathbf{x}_0\}$ is semi-eutactic, and by symmetry $S(D_n)$ is redundantly semi-eutactic. Now let $Q = \sum_{\mathbf{x} \in S(D_n)} v_{\mathbf{x}} P_{\mathbf{x}}$. If $v_{\pm 2^{-1/2}(1, 1, 0, 0, \dots, 0)} = 0$, then $Q_{12} = -v_{2^{-1/2}(1, -1, 0, 0, \dots, 0)}$. Therefore, there is no set of all-positive eutaxy coefficients for the configuration $S(D_n) \setminus \{\pm 2^{-1/2}(1, 1, 0, 0, \dots, 0)\}$ and $S(D_n)$ is not redundantly extreme.

For the lattices $\Lambda = E_7, E_8$, and Λ_{24} , it is enough to note that they contain as sublattices of equal minimal norm the root lattices $\Lambda' = A_7, D_8$, and D_{24} , respectively, which are themselves extreme. Since Id is already in the interior of the cone $\{\sum v_{\mathbf{x}} P_{\mathbf{x}} : \mathbf{x} \in S(\Lambda')\}$, it is also in the interior of $\{\sum v_{\mathbf{x}} P_{\mathbf{x}} : \mathbf{x} \in S(\Lambda) \setminus \{\pm \mathbf{x}_0\}\}$, when $\mathbf{x}_0 \in S(\Lambda) \setminus S(\Lambda')$, but by symmetry, the same is true for all $\mathbf{x}_0 \in S(\Lambda)$. For the lattice E_6 , a direct calculation for a particular choice $\mathbf{x}_0 \in S(E_6)$, which we omit, shows that $\text{Id} = \sum_{\mathbf{x} \in S} v_{\mathbf{x}} P_{\mathbf{x}}$, where $v_{\mathbf{x}} = \frac{1}{15}, \frac{1}{10}, 0$ respectively when $\langle \mathbf{x}, \mathbf{x}_0 \rangle = 0, \pm \frac{1}{2}, \pm 1$, so $S(E_6) \setminus \{\pm \mathbf{x}_0\}$ is eutactic. Another direct calculation shows that Sym^6 is spanned by $P_{\mathbf{x}}, \mathbf{x} \in S(E_6) \setminus \{\pm \mathbf{x}_0\}$. Therefore, the lattices E_6, E_7, E_8 , and Λ_{24} are redundantly extreme. \square

We establish a relation between eutaxy properties and the existence or non-existence of linear maps that satisfy certain conditions. For this purpose we recall the fundamental theorem of linear algebra, which in

our case implies that there exists a symmetric linear map Q such that

$$\begin{aligned}\langle \mathbf{x}, Q\mathbf{x} \rangle &= \langle P_{\mathbf{x}}, Q \rangle = a_{\mathbf{x}} \\ \text{trace } Q &= \langle \text{Id}, Q \rangle = -a_0,\end{aligned}$$

if and only if $\sum_{\mathbf{x} \in S} a_{\mathbf{x}} b_{\mathbf{x}} + a_0 b_0 = 0$ whenever the coefficients $b_0, (b_{\mathbf{x}})_{\mathbf{x} \in S}$ are a solution to the equation

$$(6) \quad \sum_{\mathbf{x} \in S} b_{\mathbf{x}} P_{\mathbf{x}} - b_0 \text{Id} = 0,$$

In other words, the space of possible a 's is the orthogonal complement of the space of possible b 's. Also recall that a subspace of \mathbb{R}^m contains a positive vector, if and only if its orthogonal complement does not contain a non-zero, non-negative vector. With these facts in mind we prove the following useful theorem:

- Theorem 1.** (i) *If S is extreme, there exists $\varepsilon > 0$ and $C > 0$ such that whenever T is a linear map, $\|T - \text{Id}\| < \varepsilon$, and $\|T\mathbf{x}\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in S$, then $\det T \geq 1 + C\|T^T T - \text{Id}\|$.*
- (ii) *If S is not extreme then for arbitrarily small $\varepsilon > 0$ there exists a linear map T satisfying $\|T - \text{Id}\| < \varepsilon$, $\|T\mathbf{x}\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in S$, and $\det T < 1$.*
- (iii) *S is semi-eutactic if and only if there exists $\varepsilon > 0$ and $C > 0$ such that whenever T is a linear map, $\|T - \text{Id}\| < \varepsilon$, and $\|T\mathbf{x}\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in S$, then $\det T > 1 - C\|T - \text{Id}\|^2$.*
- (iv) *Let S be minimally extreme with eutaxy coefficients $v_{\mathbf{x}}$ and let numbers $\omega_{\mathbf{x}}$ be given for $\mathbf{x} \in S$ such that $\omega_{\mathbf{x}} = \omega_{-\mathbf{x}}$, then there exists a symmetric linear map $Q \in \text{Sym}^n$ satisfying $\langle \mathbf{x}, Q\mathbf{x} \rangle = \omega_{\mathbf{x}}$ and $\text{trace } Q = \omega_0$ if and only if $\omega_0 = \sum_{\mathbf{x} \in S} \omega_{\mathbf{x}} v_{\mathbf{x}}$.*

Proof. (i) By eutaxy, there exist positive coefficients $b_{\mathbf{x}}$ and b_0 satisfying (6). Therefore, every symmetric map Q such that

$$(7) \quad \langle P_{\mathbf{x}}, Q \rangle \geq 0, \text{ for all } \mathbf{x} \in S$$

and $\langle \text{Id}, Q \rangle \leq 0$ must satisfy $\langle P_{\mathbf{x}}, Q \rangle = 0$ for all $\mathbf{x} \in S$, and therefore, by perfection $Q = 0$. Thus, if $Q \neq 0$ satisfies (7) then $\langle Q, \text{Id} \rangle > 0$. In fact, by compactness of the unit sphere in Sym^n , there must be a positive number $C' > 0$ such that $\langle Q, \text{Id} \rangle \geq C'\|Q\|$ whenever Q satisfies (7). Now let T be a linear map such that $\|T - \text{Id}\| < \varepsilon$ and $\|T\mathbf{x}\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in S$. Then $Q = T^T T - \text{Id}$ satisfies (7), and therefore $\text{trace } Q = \langle Q, \text{Id} \rangle \geq C'\|Q\|$. For each C' , there exists ε and C such that $\det T = \sqrt{\det(\text{Id} + Q)} \geq 1 + C\|Q\|$ whenever $\|T - \text{Id}\| < \varepsilon$.

- (ii) If S is not eutactic, there is no solution with positive coefficients to (6), and therefore there is a non-zero solution to (7) with $\langle Q, \text{Id} \rangle \leq 0$. Similarly, if S is eutactic but not perfect, any map Q in the complement of the linear span of $P_{\mathbf{x}}$, $\mathbf{x} \in S$, satisfies (7) and $\langle Q, \text{Id} \rangle = 0$. Consider the map $T_\alpha = \sqrt{\text{Id} + \alpha Q}$, where the square root indicates the unique positive-definite square root. Note that $\|T_\alpha \mathbf{x}\| \geq \|\mathbf{x}\|$, $\det T_\alpha = \sqrt{\det(1 + \alpha Q)} < (1 + \frac{\alpha}{n} \text{trace } Q)^{n/2} \leq 1$, and $\|T_\alpha - \text{Id}\|$ can be arbitrarily small.
- (iii) Suppose S is semi-eutactic, then there exists a non-zero solution to (6) with non-negative coefficients. Therefore, there is no symmetric map Q such that

$$(8) \quad \langle Q, P_{\mathbf{x}} \rangle > 0, \text{ for all } \mathbf{x} \in S$$

and $\langle Q, \text{Id} \rangle < 0$. In fact there must also be no symmetric map Q satisfying (7) and $\langle Q, \text{Id} \rangle < 0$, since $Q' = Q + \frac{1}{2n} \langle Q, \text{Id} \rangle \text{Id}$ would satisfy (8) and $\langle Q', \text{Id} \rangle < 0$. Again let T be a linear map such that $\|T - \text{Id}\| < \varepsilon$ and $\|T\mathbf{x}\| \geq \|\mathbf{x}\|$ for all $\mathbf{x} \in S$. Then $Q = T^T T - \text{Id}$ satisfies (7), and therefore $\text{trace } Q \geq 0$. If ε is sufficiently small, then $\det T = \sqrt{\det(\text{Id} + Q)} > 1 - C\|T - \text{Id}\|^2$.

If S is not semi-eutactic, then there is no non-zero solution to (6) with non-negative coefficients and therefore, there exists a symmetric map Q satisfying (8) and $\text{trace } Q < 0$. Again, consider the map $T_\alpha = \sqrt{\text{Id} + \alpha Q}$ and note that $\|T_\alpha \mathbf{x}\| > \|\mathbf{x}\|$, $\det T_\alpha = \sqrt{\det(1 + \alpha Q)} \leq (1 + \frac{\alpha}{n} \text{trace } Q)^{n/2} \leq 1 + \alpha \text{trace } Q$, and $\|T_\alpha - \text{Id}\|$ can be arbitrarily small.

- (iv) Since S is minimally extreme, there is a unique solution up to scaling to (6). Therefore, there is a symmetric map Q satisfying $\langle \mathbf{x}, Q\mathbf{x} \rangle = \langle Q, P_{\mathbf{x}} \rangle \geq \omega_{\mathbf{x}}$ and $\text{trace } Q = \langle Q, \text{Id} \rangle \leq \omega_0$ if and only if $\sum_{\mathbf{x} \in S} v_{\mathbf{x}} \omega_{\mathbf{x}} - \omega_0 = 0$. In fact, the map Q is unique and depends linearly on the variables $\omega_{\mathbf{x}}$.

□

Corollary 1. (Voronoi) Λ locally minimizes $d(\Lambda)$ among admissible lattices for B^n if and only if $m(\Lambda) = 1$ and Λ is extreme.

Proof. Λ is a local minimum among admissible lattices for B^n if and only if there exists $\varepsilon > 0$ such that whenever $\|T - \text{Id}\| < \varepsilon$ and $\|T\mathbf{x}\| \geq 1$ for all $\mathbf{x} \in S(\Lambda)$ then $d(T\Lambda) \geq d(\Lambda)$ [9]. □

Corollary 2. If Λ is extreme and $m(\Lambda) = 1$ then there exists $\varepsilon > 0$ and $C > 0$, such that whenever Λ' is admissible for B^n and $\delta(\Lambda', \Lambda) < \varepsilon$ then $d(\Lambda') - d(\Lambda) \geq C \min_U \delta(\Lambda', U\Lambda)$, where the minimum is over all orthogonal maps U .

Proof. If $\Lambda' = T\Lambda$ then $\min_U \delta(\Lambda', U\Lambda) \leq \delta(\sqrt{T^T T}\Lambda, \Lambda) \leq \|\sqrt{T^T T} - \text{Id}\| \leq C\|T^T T - \text{Id}\|$. \square

A consequence of the last corollary, since there can be only finitely many (up to rotation) local minima of the determinant, is that there exists $\varepsilon > 0$ and $C > 0$ such that if Λ is admissible for B^n and $d(\Lambda) - d_{B^n} < \varepsilon$ then there exists a lattice Λ' , critical for B^n , such that $\delta(\Lambda, \Lambda') < C(d(\Lambda) - d_{B^n})$. Also, if Λ is critical for a nearly spherical body K satisfying $(1 - \varepsilon')B^n \subseteq K \subseteq (1 + \varepsilon')B^n$, where $\varepsilon' < \varepsilon$ then again there exists a lattice Λ' , critical for B^n such that $\delta(\Lambda, \Lambda') < C\varepsilon'$.

Theorem 2. *Suppose that Λ is the unique critical lattice of B^n up to rotations. Then B^n is reducible if and only if Λ is redundantly extreme.*

Proof. Consider the ε -symmetrically truncated ball $B_\varepsilon = \{\mathbf{x} \in B^n : -1 + \varepsilon \leq \langle \mathbf{x}, \mathbf{p} \rangle \leq 1 - \varepsilon\}$, where $\mathbf{p} \in S^{n-1}$ is some arbitrarily chosen pole. First assume that Λ is not redundantly extreme. That is, we assume that there exists $\mathbf{x}_0 \in S(\Lambda)$ such that $S(\Lambda) \setminus \{\pm\mathbf{x}_0\}$ is not extreme. We are free to assume that Λ is rotated so that $\mathbf{x}_0 = \mathbf{p}$. Then by Theorem 1 (ii), there exists a linear map T satisfying $\|T\mathbf{x}\| \geq 1$ for all $\mathbf{x} \in S(\Lambda) \setminus \{\pm\mathbf{p}\}$, $\det T < 1$, and $\|T - \text{Id}\|$ is arbitrarily small. In fact, if $\|T - \text{Id}\|$ is small enough, then $T\mathbf{p} \notin B_\varepsilon$ and $T\Lambda$ is admissible for B_ε . Since $d(T\Lambda) < d(\Lambda)$, and since for each proper $K \subset B^n$, there exists $\varepsilon > 0$ such that $K \subset B_\varepsilon \subset B^n$, it follows that B^n is irreducible.

Now suppose that Λ is redundantly extreme. Let $T\Lambda$ be a critical lattice of B_ε . $\|T - \text{Id}\|$ can be made arbitrarily small by choosing ε sufficiently small and appropriately rotating Λ . If ε is small enough, then because $T\Lambda$ is admissible for B_ε , of the vectors of $TS(\Lambda)$ only one pair $\pm T\mathbf{x}_0$ can be within the interior of the unit ball B^n . Since Λ is redundantly extreme, the requirement that $\|T\mathbf{x}\| \geq 1$ whenever $\mathbf{x} \in S(\Lambda) \setminus \{\pm\mathbf{x}_0\}$, necessarily implies, when $\|T - \text{Id}\|$ is small enough, that $\det T \geq 1$. Of course, since $d_{B_\varepsilon} \leq d_{B^n}$, we have that $\det T = 1$ and B^n is reducible. \square

Corollary 3. *For $n = 6, 7, 8$, or 24 , the unit ball B^n is reducible.*

Proof. Recall that E_6, E_7, E_8 , and Λ_{24} are the unique critical lattices for B^n for these values of n . Recall also from Lemma 1 that they are redundantly extreme. \square

Theorem 3. *If Λ is the unique critical lattice of B^n up to rotations and Λ is redundantly semi-eutactic then B^n is not locally pessimal for packing.*

Proof. Let us call the convex hull of $B_{1/100}$ and $(1 - \varepsilon)B^n$, the ε -shaved ball B'_ε . Let $T\Lambda$ be a critical lattice of B'_ε . Again, if ε is small enough,

then of the vectors of $TS(\Lambda)$ only one pair $\pm T\mathbf{x}_0$ can be within the interior of the unit ball B^n . Since Λ is redundantly semi-eutactic, the requirement that $\|T\mathbf{x}\| \geq 1$ whenever $\mathbf{x} \in S(\Lambda) \setminus \{\pm\mathbf{x}_0\}$, necessarily implies, when $\|T - \text{Id}\|$ is small enough, that $\det T > 1 - C\|T - \text{Id}\|^2$. However, note that $\|T - \text{Id}\| < C'\varepsilon$ and $\text{vol } B'_\varepsilon / \text{vol } B^n < 1 - c\varepsilon$. Therefore, for all small enough ε , $\varphi(B'_\varepsilon) < \varphi(B^n)$ and the ball is not a local pessimum. \square

Corollary 4. *For $n = 4$ or 5 , the unit ball B^n is irreducible but not locally pessimal.*

Proof. Recall that D_4 and D_5 are the unique critical lattices for B^n for these values of n . Recall also from Lemma 1 that they are redundantly semi-eutactic but not redundantly extreme. Therefore, as a corollary of the two preceding theorems, B^n in these dimensions is irreducible but not locally pessimal. \square

Of the dimensions where the optimal packing fraction of the ball is known, the theorems given and the known results for $n = 2$ leave only the case $n = 3$ unresolved, where the unique critical lattice D_3 is minimally extreme. We show next that the critical determinant of a nearly spherical body in a dimension where the unique critical lattice is minimally extreme (as is also the case for $n = 2$, but presumably not in any other dimension) is bounded from above in first order by the average of its support function (or its radial function) evaluated at the minimal vectors of the critical lattice of the ball, weighted by the eutaxy coefficients. The form of the error term depends on whether the support function or the radial function is used.

Theorem 4. *Suppose that Λ_0 is the unique critical lattice of B^n , and Λ_0 is minimally extreme with eutaxy coefficients $v_{\mathbf{x}}$ for $\mathbf{x} \in S(\Lambda_0)$.*

(i) *Let $h_{\mathbf{x}} = 1 + \eta_{\mathbf{x}} = h_K(\mathbf{x})$ be the values of the support function of a nearly spherical body $(1 - \varepsilon)B \subseteq K \subseteq (1 + \varepsilon)B$ evaluated at $\mathbf{x} \in S(\Lambda_0)$. Then for small enough ε there is a lattice Λ , admissible for K , whose determinant $d(\Lambda)$ satisfies*

$$(9) \quad \frac{d(\Lambda)}{d(\Lambda_0)} \leq 1 + \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \eta_{\mathbf{x}} + C \max_{\mathbf{x} \in S(\Lambda_0)} |\eta_{\mathbf{x}}|^2.$$

(ii) *Let $r_{\mathbf{x}} = 1 + \rho_{\mathbf{x}} = r_K(\mathbf{x})$ be the values of the radial function of a nearly spherical body $(1 - \varepsilon)B \subseteq K \subseteq (1 + \varepsilon)B$ evaluated at $\mathbf{x} \in S(\Lambda_0)$. Then for small enough ε there is a lattice Λ' ,*

admissible for K , whose determinant $d(\Lambda')$ satisfies

$$\frac{d(\Lambda')}{d(\Lambda_0)} \leq 1 + \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \rho_{\mathbf{x}} + \varepsilon' \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|,$$

where ε' depends on ε and becomes arbitrarily small as $\varepsilon \rightarrow 0$.

Proof. (i) By Theorem 1 (iv), there exists a symmetric linear map Q such that $\langle \mathbf{x}, Q\mathbf{x} \rangle = \eta_{\mathbf{x}}$ and $\text{trace } Q = \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \eta_{\mathbf{x}}$ and this map depends linearly on the variables $\eta_{\mathbf{x}}$. Consider then the lattice $(\text{Id} + Q)\Lambda$. For all $\mathbf{x} \in S(\Lambda_0)$, the vector $(\text{Id} + Q)\mathbf{x}$ satisfies $\langle (\text{Id} + Q)\mathbf{x}, \mathbf{x} \rangle \geq h_K(\mathbf{x})$ and therefore lies outside the interior of K . Therefore Λ is admissible for K . Because Q depends linearly on $\eta_{\mathbf{x}}$, there is a constant C such that (9) holds.

(ii) The situation in this case is similar to the situation in the previous case, except we must use the values of the radial function in the directions $\mathbf{x} \in S(\Lambda_0)$, instead of the support function values, to construct an admissible lattice. By the construction given in the previous case, we can construct a lattice $\Lambda = T\Lambda_0$, in general not admissible, such that $\langle \mathbf{x}, T\mathbf{x} \rangle = 1 + \rho_{\mathbf{x}}$ and

$$\frac{d(\Lambda)}{d(\Lambda_0)} \leq 1 + \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \rho_{\mathbf{x}} + C \max_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|^2.$$

We wish to dilate this lattice by a factor $1 + \alpha$ so that $\Lambda' = (1 + \alpha)\Lambda$ is admissible. Therefore for all $\mathbf{x} \in S(\Lambda_0)$ we must have

$$\alpha \geq \alpha_{\mathbf{x}} = \frac{r_K(T\mathbf{x}/\|T\mathbf{x}\|) - \|T\mathbf{x}\|}{\|T\mathbf{x}\|}.$$

As we only want to use information about the radial function evaluated at $S(\Lambda_0)$ (and the fact that the radial function is bounded between $1 - \varepsilon$ and $1 + \varepsilon$), we bound the dilation factor as illustrated in Figure 1. In the plane containing the origin O , $(1 + \rho_{\mathbf{x}})\mathbf{x}$ (denoted A in the figure), and $T\mathbf{x}$ (denoted B), we draw the tangent AX from A to the circle of radius $1 - \varepsilon$ about the origin in the direction away from B . Note that B lies on the line through A perpendicular to OA . Since $\rho_{\mathbf{x}} < \varepsilon$, the angle $\beta = \widehat{AOX}$ satisfies $\beta \leq \arccos \frac{1-\varepsilon}{1+\varepsilon} \leq 2\sqrt{\varepsilon}$. By convexity, the continuation of the tangent from A away from the circle must lie outside of K . We mark the intersection of the tangent and the ray OB as C . Then either $\alpha_{\mathbf{x}} \leq 0$, or the boundary of K intersects the ray OB between C and B . Since T depends linearly on $\rho_{\mathbf{x}}$, the angle $\gamma = \widehat{AOB}$ satisfies $\gamma \leq C \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|$ for some constant C . By the law of

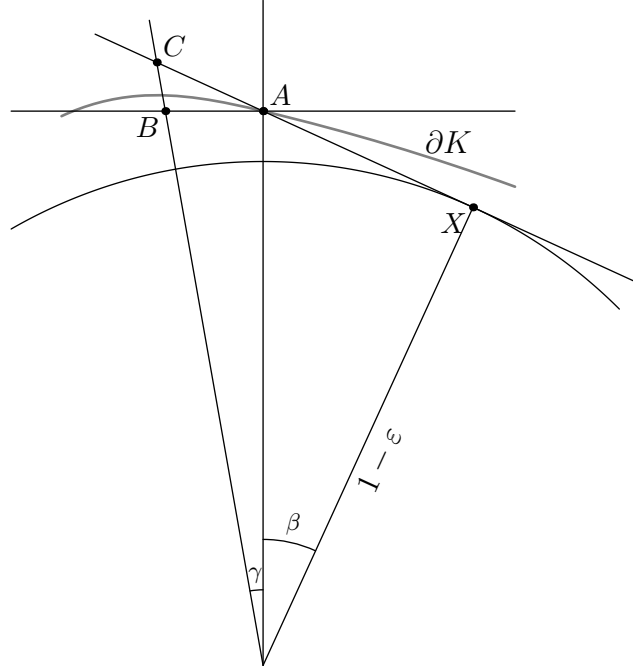


FIGURE 1. Illustration of the construction given in the proof of Theorem 4 to bound the dilation factor needed to ensure that the original point B when dilated to C lies outside the body K .

sines, we have

$$|BC| = \frac{|AB| \sin(\beta)}{\cos(\gamma + \beta)} \leq \frac{(1 + \varepsilon)\gamma\beta}{1 - \frac{1}{2}(\beta + \gamma)^2} \leq (1 - \varepsilon)\varepsilon' \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|,$$

where ε' depends on ε and becomes arbitrarily small as $\varepsilon \rightarrow 0$. Therefore, if we let $\alpha = \varepsilon' \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|$, we have that $\alpha \geq \alpha_{\mathbf{x}}$ and $(1 + \alpha)T\mathbf{x}$ is guaranteed to lie outside the interior of K for all $\mathbf{x} \in S(\Lambda_0)$.

The determinant of the dilated lattice $d(\Lambda')$ satisfies

$$\begin{aligned} \frac{d(\Lambda')}{d(\Lambda_0)} &= (1 + \alpha)^n \frac{d(\Lambda)}{d(\Lambda_0)} \\ &\leq (1 + \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \rho_{\mathbf{x}} + C \max_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|^2) (1 + \varepsilon' \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|)^n \\ &\leq 1 + \sum_{\mathbf{x} \in S(\Lambda_0)} v_{\mathbf{x}} \rho_{\mathbf{x}} + \varepsilon' \sum_{\mathbf{x} \in S(\Lambda_0)} |\rho_{\mathbf{x}}|, \end{aligned}$$

where the quadratic and higher order terms have been absorbed into the last term.

□

4. THE CASE $n = 3$

It is known that the disk B^2 is not a local pessimum [8]. Therefore, of the dimensions where the densest lattice packing of balls is known, Corollaries 3 and 4 leave only the case $n = 3$ to be resolved. Since in all the other dimensions we have seen that B^n is not a local pessimum, it may come as somewhat of a surprise, in spite of Ulam's conjecture and the intuitive argument in the introduction, that B^3 , as we will prove in this section, turns out to be a local pessimum.

The main idea of the proof comes from the fact that, as a consequence of Theorem 4, the critical determinant of a nearly spherical body is, in first order, determined by the sum $\sum_{\mathbf{x} \in S(D_3)} [r(\mathbf{x}) - 1]$ of the change (compared to the sphere) in the radial function evaluated at the twelve minimal vectors of D_3 , oriented in such a way as to minimize this sum. The average of this sum over all orientations of the minimal vectors is proportional to the average change in the radial function over the sphere, which is in turn proportional in first order to the change in volume of the body. If the sum above is not a constant independent of the orientation of the minimal vectors, then at some orientation it is smaller than the average and the optimal packing fraction is increased in first order. It is not hard to show that any even function for which the sum above is independent of the orientation must be a second degree spherical harmonic up to a constant. In first order, such a change to the radial function corresponds simply to a linear transformation. Since we know that the packing fraction is invariant with respect to linear transformations, we may pick a linear transformation that eliminates the second degree term of the harmonic expansion.

Any even continuous function f on S^2 can be expanded in terms of spherical harmonics $f(\mathbf{x}) = \sum_{l=0, l \text{ even}}^{\infty} f_l(\mathbf{x})$, where $f_l(\mathbf{x})$ is a homogeneous harmonic polynomial of degree l in $\mathbf{x} \in \mathbb{R}^3$ restricted to S^2 , and the series converges at least in $L^2(\sigma)$ (let σ denote the rotation-invariant probability measure on S^2). We may also write $f_l = \pi_l f$, where π_l is the orthogonal projection from $L^2(\sigma)$ to the finite dimensional space of spherical harmonics of degree l .

Lemma 2. *Given $\varepsilon > 0$, there exists $\varepsilon' > 0$ such that if $K \subset \mathbb{R}^3$ satisfies $(1 - \varepsilon')B^3 \subseteq K \subseteq (1 + \varepsilon')B^3$, then K has a linear image $K' = TK$ that satisfies $(1 - \varepsilon)B^3 \subseteq K' \subseteq (1 + \varepsilon)B^3$ and whose radial function has mean 1 and vanishing second spherical harmonic component.*

Proof. Let Sym_0^3 be the space of zero-trace symmetric linear maps $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ equipped with Hilbert-Schmidt norm. We identify Sym_0^3 also with the space of zero-trace quadratic forms $\mathbb{R}^3 \rightarrow \mathbb{R}$. Consider the map $F_K : \text{Sym}_0^3 \rightarrow \text{Sym}_0^3$, given by $[F_K(A)](\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle - \pi_2[r_{(1+A)K}](\mathbf{x})$, where A is viewed as a linear map and $F_K(A)$ is viewed as a quadratic form. Note that the space of second degree spherical harmonics is the same as the space of zero-trace quadratic forms.

When K is the unit ball B^3 , we have

$$r_{(1+A)B^3}(\mathbf{x}) = \frac{1}{\|(1+A)^{-1}\mathbf{x}\|} = 1 + \langle \mathbf{x}, A\mathbf{x} \rangle + O(\|A\|^2), \quad (\mathbf{x} \in S^2)$$

When taking the second harmonic component of the above, the constant 1 vanishes, the second term is preserved, and the third term contributes a term that is again of order at most $\|A\|^2$. Therefore, $\|F_{B^3}(A)\| < C\|A\|^2$. Fix a small closed ball $\mathcal{B} \subseteq \text{Sym}_0^3$ around 0 so that $F_{B^3}(\mathcal{B}) \subseteq \frac{1}{2}\mathcal{B}$. Note that $F_K(A)$ can be made arbitrarily close to $F_{B^3}(A)$ uniformly with respect to $A \in \mathcal{B}$ by appropriately choosing ε' . Then if ε' is small enough, F_K maps \mathcal{B} into \mathcal{B} , and by Brouwer's fixed point theorem there is a fixed point $F_K(A) = A$. Hence, the second spherical harmonic component of $r_{(1+A)K}$ vanishes. The mean of $r_{(1+A)K}$ might not be 1, but all that is left to do is to contract or dilate $(1+A)K$, and this operation does not change the fact that the second spherical harmonic component vanishes. By restricting ε' , the norm of A can be made as small as needed to guarantee that $(1-\varepsilon)B^3 \subseteq K' \subseteq (1+\varepsilon)B^3$. \square

Let \mathbf{p} represent a chosen north pole on S^2 . We say that a function or measure on S^2 is *zonal* if it is invariant under all rotations that preserve \mathbf{p} . If $P_l(t)$ is the Legendre polynomial of degree l , then $h_l(\mathbf{x}) = P_l(\langle \mathbf{x}, \mathbf{p} \rangle)$ is the unique zonal harmonic of degree l whose value at \mathbf{p} is 1. Given a function f and a zonal measure μ on S^2 , we can define their convolution $(\mu * f)(\mathbf{y}) = \int_{S^2} f(\mathbf{x}) d\mu(U_{\mathbf{y}}(\mathbf{x}))$, where $U_{\mathbf{y}}$ is any rotation that maps \mathbf{y} to \mathbf{p} . Such a convolution satisfies Young's inequality, and in particular as a special case $\|\mu * f\|_1 \leq \mu(S^2)\|f\|_1$. Convolution with a zonal measure acts as a multiplier transformation on the harmonic expansion, i.e., $(\mu * f)(\mathbf{x}) = \sum_{l=0}^{\infty} c_l f_l(\mathbf{x})$. The multiplier coefficients can be found by performing the convolution on the zonal harmonic $c_l = (\mu * h_l)(\mathbf{p}) = \int_{S^2} h_l(\mathbf{x}) d\mu(\mathbf{x})$ [12].

We now go on to prove two lemmas related to the configuration of minimal vectors in D_3 . Let D_3 be rotated so that \mathbf{p} is one of the minimal vectors. Then there is a unique zonal measure μ such that for

every continuous zonal function f ,

$$\int_{S^2} f(\mathbf{y}) d\mu(\mathbf{y}) = \frac{1}{2} \sum_{\mathbf{x} \in S(D_3)} f(\mathbf{x}).$$

Since $\langle \mathbf{x}, \mathbf{p} \rangle = 1, \frac{1}{2}, 0, -\frac{1}{2}, -1$ with multiplicities 1, 4, 2, 4, 1 as \mathbf{x} ranges over $S(D_3)$, we have that μ is the zonal measure with total weight 6, distributed as follows: weight $1/2$ on each of the north and south poles, weight 2 on each line of latitude 30° north and south, and weight 1 on the equator.

Lemma 3. *Let $c_l = P_l(1) + 4P_l(\frac{1}{2}) + P_l(0) = 1 + 4P_l(\frac{1}{2}) + P_l(0)$. Then $c_l = 0$ if and only if $l = 2$. Moreover, $|c_l - 1| < Cl^{-1/2}$ for some constant C .*

Proof. It would here be advantageous to work with the rescaled Legendre polynomials $Q_l(t) = 2^l P_l(t)$. Their recurrence relation is given by $Q_{l+1}(t) = 2(2l+1)tQ_l(t) - 4l^2Q_{l-1}(t)$. It is clear from the recurrence relation and the base cases $Q_0(t) = 1$ and $Q_1(t) = 2t$ that the values of $Q_l(t)$ at $t = 0, \frac{1}{2}, 1$ are integers. We are interested in their residues modulo 8. For $t = 0$, $Q_{l+2}(0)/2^{l+2} = -(l+1)^2Q_l(0)/2^l$, and so by induction $Q_l(0)$ is divisible by 2^l . For $t = \frac{1}{2}$, $Q_{l+1}(\frac{1}{2})$ is odd whenever $Q_l(\frac{1}{2})$ is odd, and hence by induction for all l . For $t = 1$, $Q_l(1) = 2^l P_l(1) = 2^l l!$ is also divisible by 2^l . Therefore, for all $l \geq 3$ we have $2^l c_l = Q_l(0) + 4Q_l(\frac{1}{2}) + Q_l(1) \equiv 4 \pmod{8}$ and c_l cannot vanish. For $l = 2$, we have $P_l(t) = \frac{3}{2}t^2 - \frac{1}{2}$ and $c_l = 1 + 4 \cdot (-\frac{1}{8}) - \frac{1}{2} = 0$. The second part of the lemma follows from the bound $|P_l(t)| < (\pi l \sqrt{1-t^2}/2)^{-1/2}$ [13]. \square

Lemma 4. *Let μ be the zonal measure described above and let Φ be the operator of convolution with μ . Then its multiplier coefficients are $c_l = 1 + 4P_l(\frac{1}{2}) + P_l(0)$. Further, let Z be the space, equipped with the $L^1(\sigma)$ norm, of even functions f on S^2 for which $f_2 = 0$. Then Φ maps Z to Z , and as an operator $Z \rightarrow Z$ it is one-to-one, bounded, and has a bounded inverse.*

Proof. As noted above, the multiplier coefficients of a convolution are given by $c_l = \int_{S^2} P_l(\langle \mathbf{x}, \mathbf{p} \rangle) d\mu(\mathbf{x}) = 1 + 4P_l(\frac{1}{2}) + P_l(0)$. As a consequence of Lemma 3, Φ clearly maps Z to Z and is one-to-one on Z . Since Φ is a convolution operator with a finite measure it is a bounded operator $Z \rightarrow Z$.

We now construct an operator $\Psi : Z \rightarrow Z$, which we then show to be the inverse of Φ . Let

$$\Psi = \sum_{k=0}^3 (\text{Id} - \Phi)^k + \sum_{l \text{ even}, l \neq 2} c_l^{-1} (1 - c_l)^4 \pi_l.$$

The norm of π_l induced by the L^1 norm does not exceed the L^1 norm of the projection kernel $K_l(\mathbf{x}, \cdot)$, but $\|K_l(\mathbf{x}, \cdot)\|_1 \leq \|K_l(\mathbf{x}, \cdot)\|_2 = \sqrt{2l+1}$. Therefore, the sum $\sum_{l \text{ even}, l \neq 2} |c_l^{-1}| \cdot |1 - c_l|^4 \cdot \|\pi_l\|$ converges and so Ψ is bounded. Note also that Ψ is a multiplier transform and its multiplier coefficients are simply c_l^{-1} for all even $l \neq 2$. Therefore $\Psi = \Phi^{-1}$. \square

We can now prove the main result.

Theorem 5. *There exists $\varepsilon > 0$ such that if $K \subset \mathbb{R}^3$ is a non-ellipsoidal origin-symmetric convex solid and $(1 - \varepsilon)B^3 \subseteq K \subseteq (1 + \varepsilon)B^3$, then $\varphi(K) > \varphi(B^3)$. In other words, B^3 is a local pessimum for packing.*

Proof. Given Lemma 2 and the fact that $\varphi(K)$ is invariant under linear transformations of K , we may assume without loss of generality that K is a non-spherical solid whose radial function has an expansion in spherical harmonics of the form

$$r_K(\mathbf{x}) = 1 + \rho(\mathbf{x}) = 1 + \sum_{l \text{ even}, l \geq 4} \rho_l(\mathbf{x}).$$

Therefore, the volume of K satisfies

$$\text{vol } K = \frac{4\pi}{3} \int_{S^2} r_K^3 d\sigma \geq \frac{4\pi}{3} \left(\int_{S^2} r_K d\sigma \right)^3 = \frac{4\pi}{3}.$$

We consider all the rotations $U(K)$ of the solid K and the determinants of the admissible lattices obtained when the construction of Theorem 4 (ii) is applied to $U(K)$. Note that the determinants obtained depend only on $\rho(U^{-1}(\mathbf{x}))$ for $\mathbf{x} \in S(D_3)$. Let us define $\Delta_K = \frac{\phi(K)^{-1}}{\phi(B^n)^{-1}} - 1$. Combining our bound on $\text{vol } K$ with Theorem 4 (ii) we get

$$(10) \quad \Delta_K \leq \min_{U \in SO(3)} \left[\frac{1}{4} \sum_{\mathbf{x} \in S(D_3)} \rho(U^{-1}(\mathbf{x})) + \varepsilon' \sum_{\mathbf{x} \in S(D_3)} |\rho(U^{-1}(\mathbf{x}))| \right].$$

We may pick a single point $\mathbf{x}_0 \in S(D_3)$ and decompose $SO(3)$ into subsets $\mathcal{U}_{\mathbf{y}}$ of all rotations such that $U^{-1}(\mathbf{x}_0) = \mathbf{y}$. In each subset $\mathcal{U}_{\mathbf{y}}$ the minimum on the right hand side of (10) is no larger than the

average value over $\mathcal{U}_{\mathbf{y}}$ (with respect to the obvious uniform measure). This averaging procedure transforms (10) into

$$(11) \quad \Delta_K \leq \min_{\mathbf{y} \in S^2} \left[\frac{1}{2} \Phi[\rho](\mathbf{y}) + 2\varepsilon' \Phi[|\rho|](\mathbf{y}) \right],$$

where Φ is the same convolution operator as in Lemma 4.

For integrable functions f and g over a domain of measure 1, such that f is of zero mean and g is non-negative, we have $\min[f+g] \leq -\|\min(f+g, 0)\|_1 \leq \|g\|_1 - \|\min(f, 0)\|_1$. Since $\|f\|_1 = 2\|\min(f, 0)\|_1$, we have $\min[f+g] \leq -\frac{1}{2}\|f\|_1 + \|g\|_1$. Therefore, if we let $f = \frac{1}{2}\Phi[\rho](\mathbf{y})$ and let $g = \varepsilon'\Phi[|\rho|](\mathbf{y})$, we obtain

$$\Delta_K \leq -\frac{1}{4}\|\Phi^{-1}\|^{-1} \cdot \|\rho\|_1 + 2\varepsilon'\|\Phi\| \cdot \|\rho\|_1.$$

Since ε' can be made as small as needed by decreasing ε we conclude that there is a coefficient $c > 0$ such that $\Delta_K \leq -c\|\rho\|_1$. \square

Remark 1. *Since the optimal lattice packing fraction of B^3 is also its optimal packing fraction for non-lattice packings [7], and since any ellipsoid can be packed at a higher packing fraction when not restricted to a lattice [3], it follows from the theorem that B^3 is also a local pessimum among origin-symmetric solids with respect to packing that allows rotations and non-lattice translations. The case of origin-non-symmetric solids remains open.*

Remark 2. *Note that the theorem provides a bound for the improvement in the packing fraction of K over B^3 which is linear in terms of the L^1 norm of the difference between the radial functions of B^3 and an appropriate linear transformation of K .*

Remark 3. *It appears that the size of the neighborhood ε that can be obtained using this method of proof is primarily governed by $\|\Phi^{-1}\|^{-1}$, which is in turn primarily governed by $\min_{l \geq 4} |c_l|$. By numerical computation, we find that $|c_l| > 0.2$ for all even $l \geq 4$, except for $l = 10$, for which $c_l \approx 10^{-3}$.*

Acknowledgments. A special acknowledgment is owed to Fedor Nazarov who contributed significantly to the work presented but declined co-authorship. In particular, Nazarov contributed crucial ideas for the proofs of Lemma 4 and part (ii) of Theorem 4, provided a simpler proof of Lemma 3 than in the original manuscript, and helped in the preparation of a previous version of the manuscript.

REFERENCES

1. H. Cohn and A. Kumar, *Optimality and uniqueness of the Leech lattice among lattices*, Ann. Math. **170** (2009), 1003.
2. J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices, and groups (third edition)*, Springer, New York, 1998.
3. A. Donev, F. H. Stillinger, P. M. Chaikin, and S. Torquato, *Unusually dense crystal packing of ellipsoids*, Phys. Rev. Lett. **92** (2004), 255506.
4. M. Gardner, *New mathematical diversions (revised edition)*, Math. Assoc. Amer., Washington, 1995.
5. ———, *The colossal book of mathematics*, Norton, New York, 2001.
6. P. M. Gruber, *Convex and discrete geometry*, Springer, New York, 2007.
7. T. C. Hales, *A proof of the Kepler conjecture*, Ann. Math. **162** (2005), 1063.
8. K. Mahler, *On the area and the densest packing of convex domains*, Proceedings of the Koninklijke Nederlandse Academie van Wetenschappen **50** (1947), 108.
9. J. Martinet, *Perfect lattices in euclidean spaces*, Springer, Berlin, 2003.
10. F. L. Nazarov, *On the Reinhardt problem of lattice packings of convex regions, local extremality of the Reinhardt octagon*, J. Soviet Math. **43** (1988), 2687.
11. K. Reinhardt, *Über die dichteste gitterförmige Lagerung kongruente Bereiche in der Ebene und eine besondere Art konvexer Kurven*, Abh. Math. Sem., Hamburg, Hansischer Universität, Hamburg **10** (1934), 216.
12. F. E. Schuster, *Convolutions and multiplier transformations of convex bodies*, Transact. Amer. Math. Soc. **359** (2007), 5567.
13. G. Szegő, *Orthogonal polynomials*, American Mathematical Society, Providence, 2003.

YOAV KALLUS, CENTER FOR THEORETICAL SCIENCE, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08544, YKALLUS@PRINCETON.EDU