



The ball as a pessimal shape for packing

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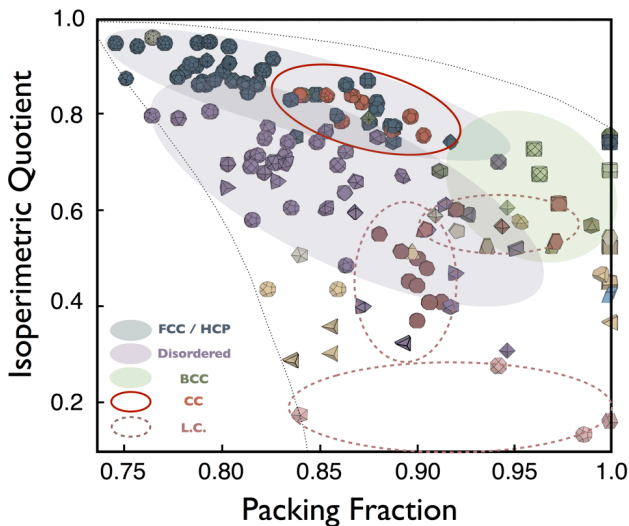


From Hilbert's 18th Problem

“How can one arrange most densely in space an infinite number of equal solids of a given form, e.g., spheres with given radii or regular tetrahedra with given edges, that is, how can one so fit them together that the ratio of the filled to the unfilled space may be as large as possible?”

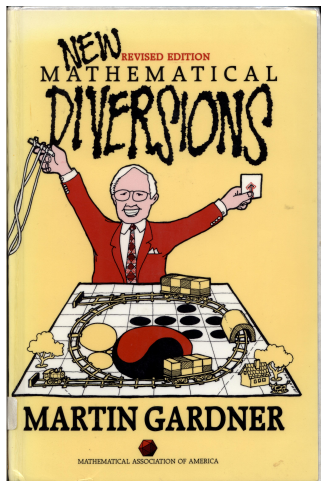


Packing convex shapes



Damasceno, Engel, and Glotzer, 2012.

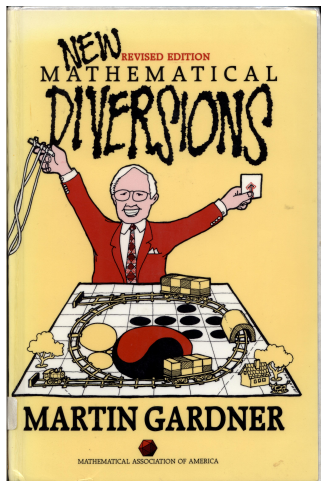
Ulam's Conjecture



“Stanislaw Ulam told me in 1972 that he suspected the sphere was the worst case of dense packing of identical convex solids, but that this would be difficult to prove.”

1995 postscript to the column “Packing Spheres”

Ulam's Last Conjecture



“Stanislaw Ulam told me in 1972 that he suspected the sphere was the worst case of dense packing of identical convex solids, but that this would be difficult to prove.”

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Modified Ulam's Conjecture

A **lattice** in \mathbb{R}^n is a linear image of the integer lattice $\Lambda = T\mathbb{Z}^n$. Its **determinant** is $d(\Lambda) = |\det T|$.

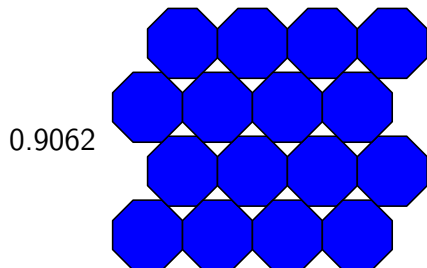
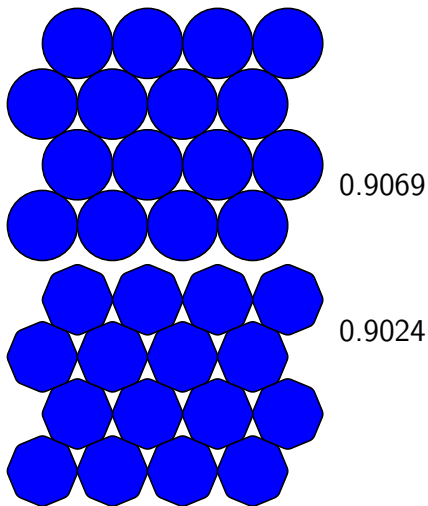
Let K be a centrally-symmetric convex body. A lattice Λ is **admissible** K if Λ intersects the interior of K at the origin only. Equivalently, $K + \mathbf{l}$ and $K + \mathbf{l}'$ have disjoint interiors for $\mathbf{l} \neq \mathbf{l}' \in 2\Lambda$, so $K + 2\Lambda$ is a **packing** of K .

The **critical determinant** of K is $d_K = \min_{\Lambda \text{ admis. } K} d(\Lambda)$. Its **lattice packing density** is $\delta_L(K) = 2^{-n}|K|/d_K$, and is affine invariant.

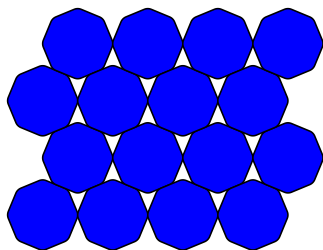
Conjecture

When $K \subseteq \mathbb{R}^3$ is not an ellipsoid, then $\delta_L(K) > \delta_L(B^3) = \pi/\sqrt{18}$

In 2D disks are not worst



Reinhardt's conjecture



0.9024

Conjecture (K. Reinhardt, 1934)

The smoothed octagon is an absolute minimum of δ_L among convex, centrally symmetric planar bodies.

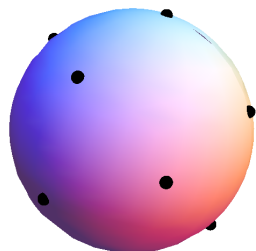
Theorem (F. Nazarov, 1986)

The smoothed octagon is a local minimum (w.r.t. Banach-Mazur distance).

K. Reinhardt, Abh. Math. Sem., Hamburg, Hansischer Universität, Hamburg 10 (1934), 216

F. Nazarov, J. Soviet Math. 43 (1988), 2687

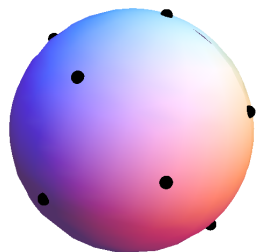
Higher dimensions



$\delta_L(B^n)$ is known for $n = 2, 3, 4, 5, 6, 7, 8,$ and 24 .

A lattice Λ that achieves a local maximum packing density is **extreme**. That is, there is ϵ s.t. when $\|T - \text{Id}\| < \epsilon$ then $\det T \geq 1$ (i.e. $d(T\Lambda) \geq d(\Lambda)$) or $\|T\mathbf{x}\| < 1$ for some $\mathbf{x} \in \partial B^n \cap \Lambda$ (i.e. $T\Lambda$ is not admissible).

Higher dimensions

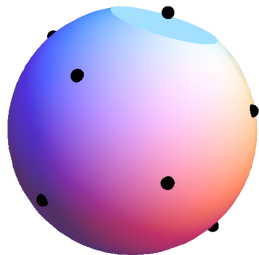


A lattice Λ that achieves a local maximum packing density is **extreme**. That is, there is ϵ s.t. when $\|T - \text{Id}\| < \epsilon$ then $\det T \geq 1$ (i.e. $d(T\Lambda) > d(\Lambda)$) or $\|T\mathbf{x}\| < 1$ for some $\mathbf{x} \in \partial B^n \cap \Lambda$ (i.e. $T\Lambda$ is not admissible).

Let $W = \partial B^n \cap \Lambda$, then Λ is extreme if and only if there exists no nonzero $Q \in \text{Sym}^n$ such that $\langle \mathbf{x}, Q\mathbf{x} \rangle \geq 0$ for all $\mathbf{x} \in W$ and $\text{trace } Q \leq 0$.

By Farkas' Lemma, Λ is extreme if and only if $\text{Id} \in \text{int cone}_{\mathbf{x} \in W} \mathbf{x} \otimes \mathbf{x}$.

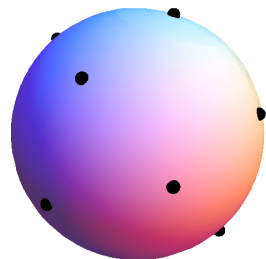
$$n = 6, 7, 8, 24$$



For $n = 6, 7, 8, 24$, the lattice Λ_n achieving $\delta_L(B^n)$ has a contact configuration W_n that is redundantly extreme: for any $W' = W_n \setminus \{\pm \mathbf{x}'\}$, we still have $\text{Id} \in \text{int cone}_{\mathbf{x} \in W'} \mathbf{x} \otimes \mathbf{x}$.

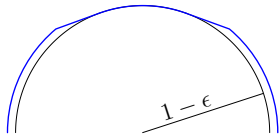
Therefore, a slightly dented sphere B' has $d_{B'} = d_{B^n}$, $|B'| < |B^n|$, so $\delta_L(B') < \delta_L(B^n)$.

$n = 4, 5$



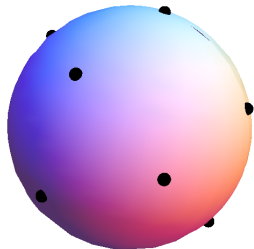
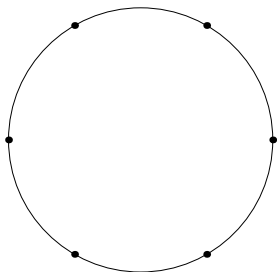
For $n = 4, 5$, the lattice Λ_n achieving $\delta_L(B^n)$ is nearly redundantly extreme: for any $W' = W_n \setminus \{\pm \mathbf{x}'\}$, we only have $\text{Id} \in \partial \text{cone}_{\mathbf{x} \in W'} \mathbf{x} \otimes \mathbf{x}$.

Therefore, there is ϵ such that when $\|T - \text{Id}\| < \epsilon$, $T\mathbf{x} \geq 1$ for all but one $\mathbf{x} \in W_n$, then $\det T > 1 - C\|T - \text{Id}\|^2$.



Consider the sphere “shaved” to a depth ϵ on two antipodal caps. Then $d_{B'} > (1 - C\epsilon^2)d_{B^n}$, $|B'| < (1 - c\epsilon)|B^n|$, and so so $\delta_L(B') < \delta_L(B^n)$.

$n = 2, 3$



For $n = 2, 3$, we have that $\{\mathbf{x} \otimes \mathbf{x} : \mathbf{x} \in W_n\}$ is a basis for Sym^n . So for any even $f : W_n \rightarrow \mathbb{R}$, there is a unique $Q \in \text{Sym}^n$ s.t. $\langle \mathbf{x}, Q\mathbf{x} \rangle = f(\mathbf{x})$.

For $f = r_K - 1$, where $(1 - \epsilon)B^n \subseteq K \subseteq (1 + \epsilon)B^n$, this gives

$$\frac{d_K}{d_{B^n}} \leq 1 + \frac{1}{2(n+1)} \sum_{\mathbf{x} \in W_n} f(\mathbf{x}) + \epsilon' \sum_{\mathbf{x} \in W_n} |f(\mathbf{x})|,$$

where $\epsilon' = o(1)$ in ϵ .

In fact, W_n can be replaced by $U(W_n)$ for any $U \in SO(n)$.

Crucial difference of $n = 2$ vs. $n = 3$

For given $f : S^{n-1} \rightarrow \mathbb{R}$, let $\Phi[f] : SO(n) \rightarrow \mathbb{R}$ be given by
$$\Phi[f](U) = \frac{1}{2(n+1)} \sum_{\mathbf{x} \in W_n} f(U(\mathbf{x})).$$

Let μ_n be the measure on S^{n-1} supported with equal weights on W_n , then $\Phi[f] = 0$ if and only if $\pi_\ell f = 0$ for all ℓ such that $\pi_\ell \mu_n \neq 0$.

For $n = 2$, $\pi_\ell \mu_n = 0$ for $\ell = 2, 4, 8, 10, 14, \dots$

For $n = 3$, $\pi_\ell \mu_n = 0$ only for $\ell = 2$.

Moreover, for $n = 3$, if $\pi_2 f = 0$, then $\|\Phi[f]\|_1 \geq c \|f\|_1$.

3-ball is locally pessimal

For $(1 - \epsilon)B^3 \subseteq K' \subseteq (1 + \epsilon)B^3$, there is always a linear map which gives $K = TK'$ such that $\pi_0 f = \pi_2 f = 0$, where $f = r_K - 1$.

$$\frac{d_K}{d_{B^3}} \leq \min_{U \in SO(3)} 1 + \Phi[f](U) + \epsilon' \Phi[|f|](U),$$

$$\frac{|K|}{|B^3|} \geq \overline{(1 + f)^3} \geq (\overline{1 + f})^3 = 1.$$

Since ϵ' is as small as wanted, then $\delta_L(K)/\delta_L(B^3) > 1 + c' \|f\|_1$.

3-ball is locally pessimal

Finally we have,

Theorem

There is $\epsilon > 0$ such that for any convex, centrally-symmetric K satisfying $(1 - \epsilon)B^3 \subseteq K \subseteq (1 + \epsilon)B^3$, we have $\delta_L(K) \geq \delta_L(B^3)$, with equality only for ellipsoids.

A similar argument works for covering

Theorem

There is $\epsilon > 0$ such that for any convex, centrally-symmetric K satisfying $(1 - \epsilon)B^3 \subseteq K \subseteq (1 + \epsilon)B^3$, we have $\vartheta_L(K) \leq \vartheta_L(B^3)$, with equality only for ellipsoids.