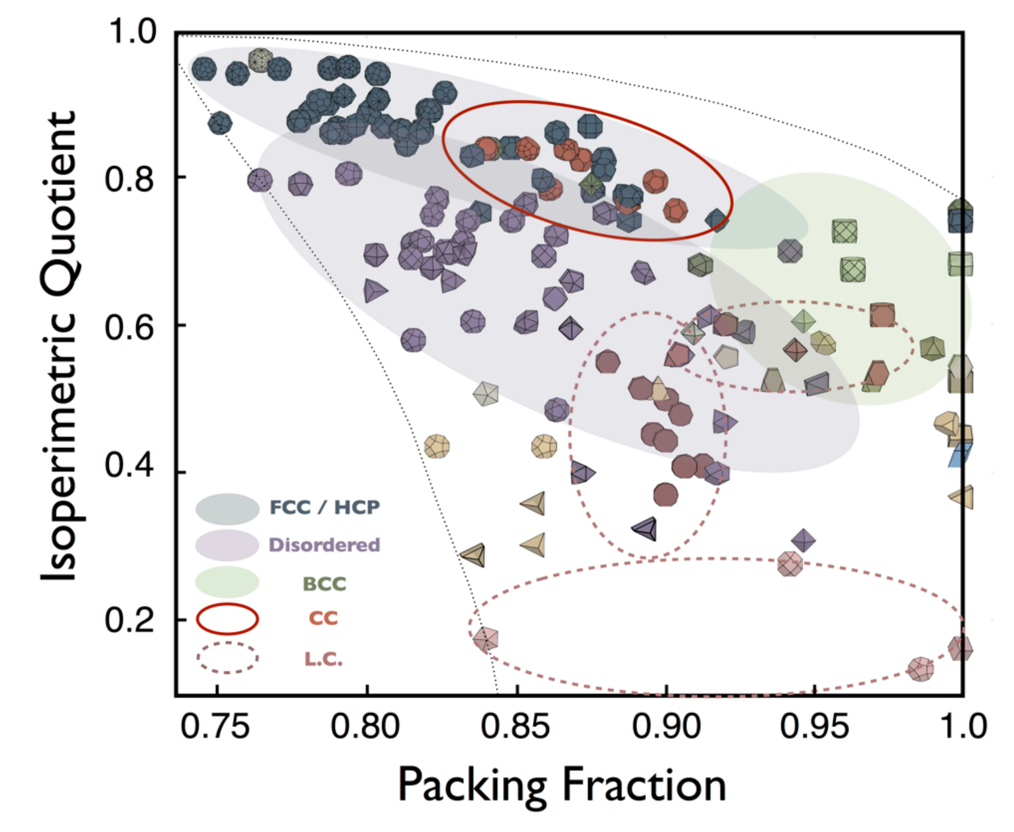
Pessimal Shapes for Packing and Covering

Yoav Kallus
Princeton University

Background

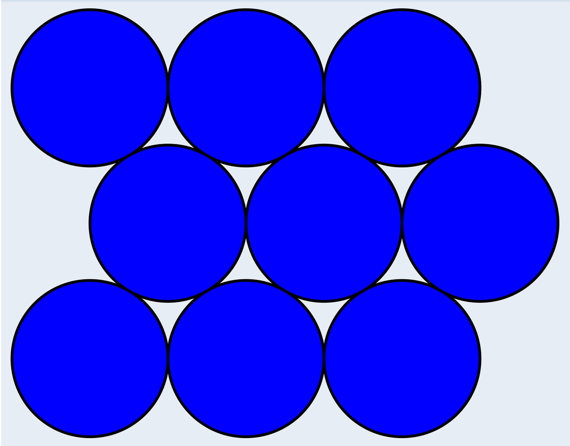
Ulam's Last Conjecture



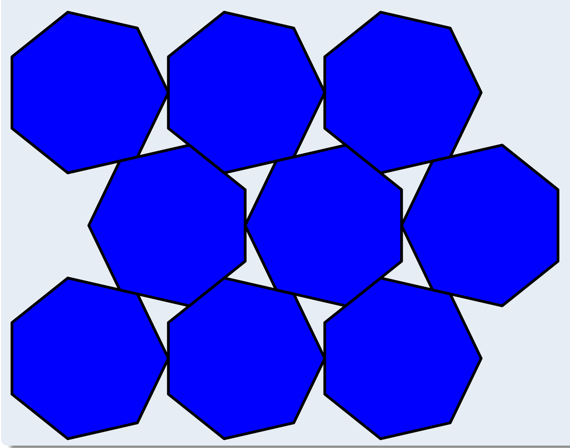
Putative optimal packing densities due to P.F. Damasceno, M. Engel, and S.C. Glotzer, 2012.

"Stanislaw Ulam told me in 1972 that he suspected the sphere was the worst case of dense packing of identical convex solids, but that this would be difficult to prove." (Martin Gardner, 1995 postscript to the column "Packing Spheres" [1])

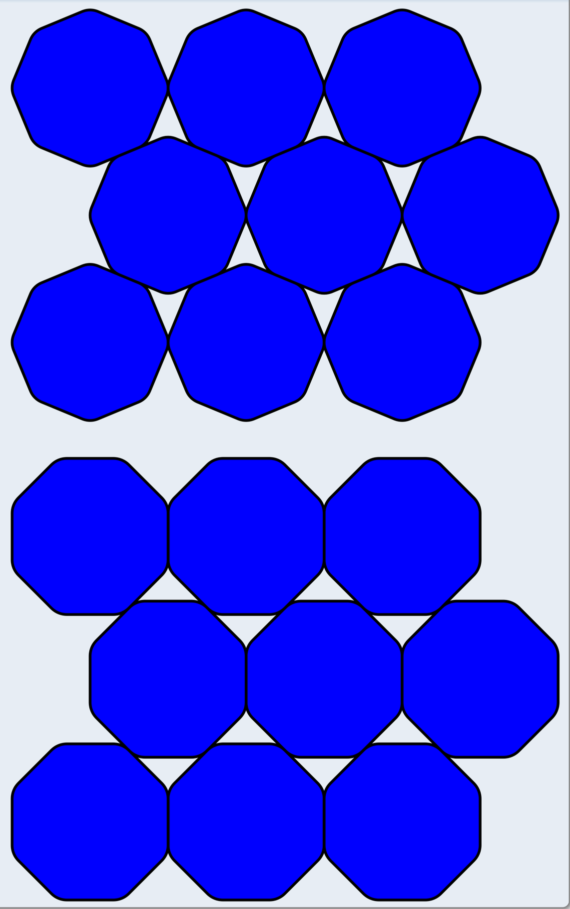
Analogous conjecture fails in 2D



Optimal packing fraction with disks: 0.9069



Optimal packing fraction with rounded octagons: 0.9024

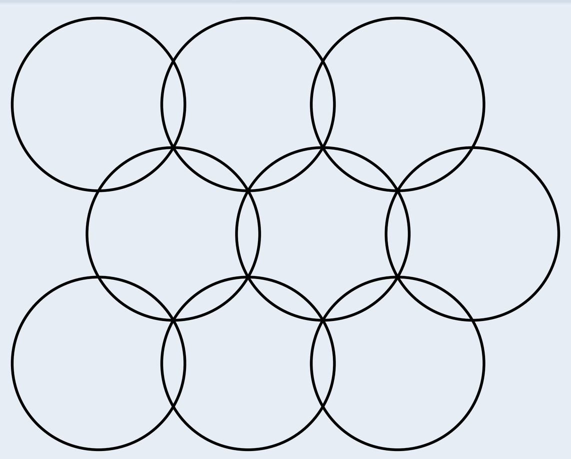


Highest known packing fraction with regular heptagons: 0.8926

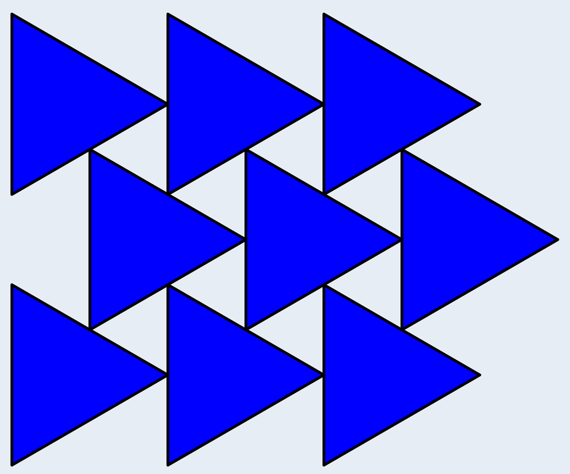
Reinhardt's Conjecture (1934)

The rounded octagon is the pessimal shape for packing in 2D among centrally symmetric convex shapes [2]. (Known to be locally pessimal [3].)

Known pessima



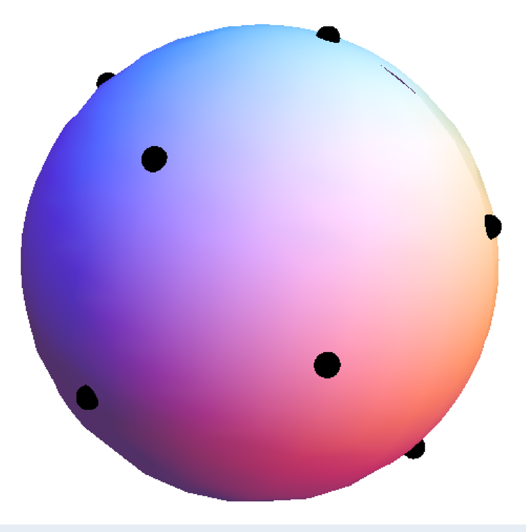
Circular disks are known to be worst for covering among centrally symmetric convex shapes in 2D [4].



Triangles are known to be worst for packing using only translations among convex shapes in 2D [5].

Theory

Optimal lattice sphere packing



Contact points in f.c.c.

- Let S be the set of contact points on a sphere at the origin in a lattice packing. A lattice Λ is *extreme* (locally optimal for packing spheres) if and only if, for $T \approx \mathrm{Id}$,

$$||T\mathbf{x}|| \geq ||\mathbf{x}|| \text{ for all } \mathbf{x} \in S(\Lambda) \implies \det T > 1.$$

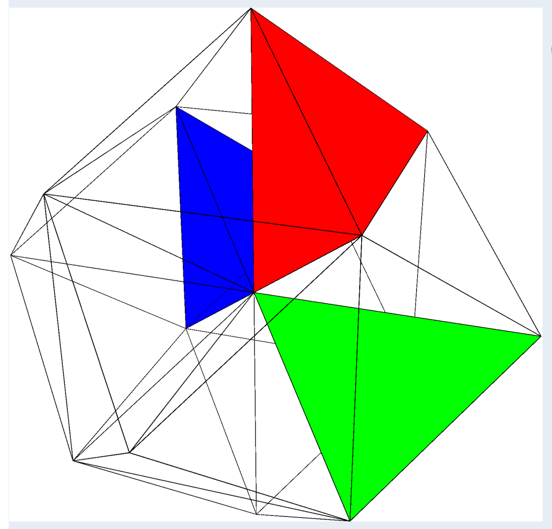
- The optimal sphere packing lattices are known for $d \leq 8$ and $d = 24$.

Eutaxy and Perfection

- For any point $\mathbf{x}$, let $P_{\mathbf{x}} = \mathbf{x}\mathbf{x}^T \in \mathrm{Sym}_d$, a symmetric $d \times d$ matrix..
- A set S is perfect if $\{P_{\mathbf{x}} : \mathbf{x} \in S\}$ spans Sym_d.
- A set S is eutactic if $\sum_{\mathbf{x}\in S} \eta_{\mathbf{x}} P_{\mathbf{x}} = \mathrm{Id}$ for some coefficients $\eta_{\mathbf{x}} > 0$.
- Theorem (Voronoi [6]): a lattice is extreme if and only if it is perfect and eutactic.

Optimal lattice sphere covering

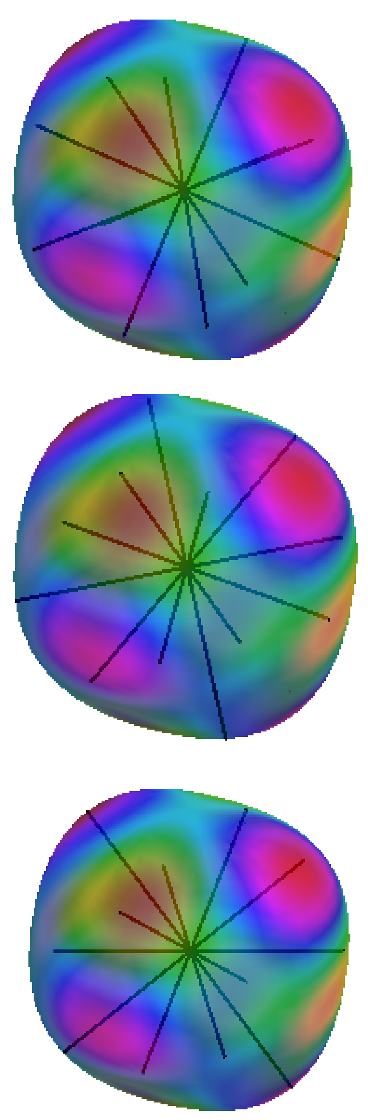
- Let T be a simplex with vertices $\mathbf{x}_0, \ldots, \mathbf{x}_d$ and circumcenter $\mathbf{c} = \sum_i \alpha_i \mathbf{x}_i$. Define $Q_T = \sum_i \alpha_i (\mathbf{x}_i - \mathbf{c})(\mathbf{x}_i - \mathbf{c})^T$.



Inequivalent Delaunay simplices in b.c.c.

- Theorem (Barnes and Dickson [7]): consider a covering of $\mathbb{R}^d$ by spheres with centers at points of a lattice with a unique Delaunay triangulation. The lattice is covering-extreme if and only it is semi-eutactic: $\sum \eta_T Q_T = \mathrm{Id}$, where the sum is over all Delaunay simplices of largest circumradius and $\eta_T \geq 0$.
- The optimal sphere covering lattices are known for $d \leq 5$.

The main lemma

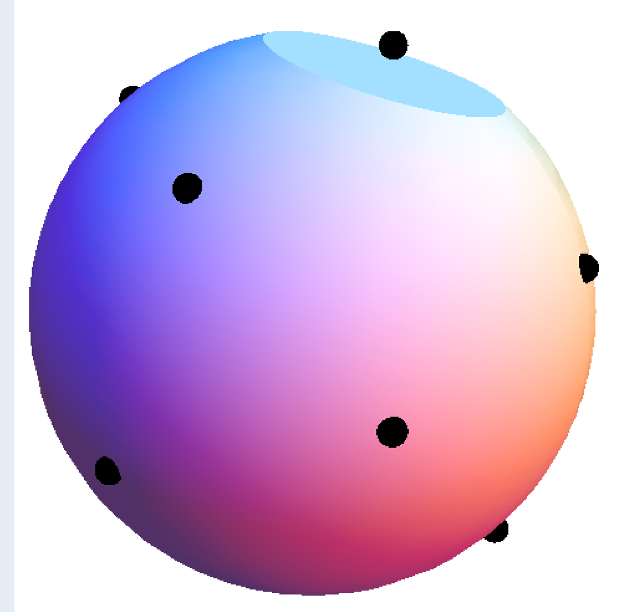


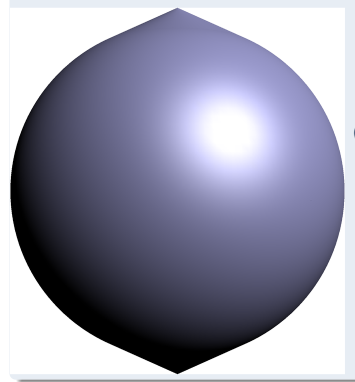
- Consider $\mathbf{x}_1, \ldots, \mathbf{x}_n$, points on the sphere S^2, such that $\sum_{i,j=1}^n P_l(\langle \mathbf{x}_i, \mathbf{x}_j \rangle) = 0$ for $l = 2$, but not for any other even l
- Example: the contact points in f.c.c.
- Example: all the vertices of Delaunay simplices in b.c.c., when the simplices are superimposed so that their circumcenters coincide.
- Let f be an even function $S^2 \to \mathbb{R}$, $R \in SO(3)$ a rotation matrix. $\sum_{i=1}^n f(R\mathbf{x}_i)$ is independent of R if and only if the expansion of $f(\mathbf{x})$ in spherical harmonics terminates at $l = 2$.

Results

Dimensions $d \geq 4$: ball not pessimal, even locally

- For packing in $d = 6, 7, 8$, and 24, the optimal lattice is redundantly extreme and so a slightly truncated ball is worse than the ball for packing in a lattice [8].
- For packing in $d = 4$ and 5, the optimal lattice is not redundantly extreme, but redundantly semi-eutactic, and the ball is still not locally pessimal [8].





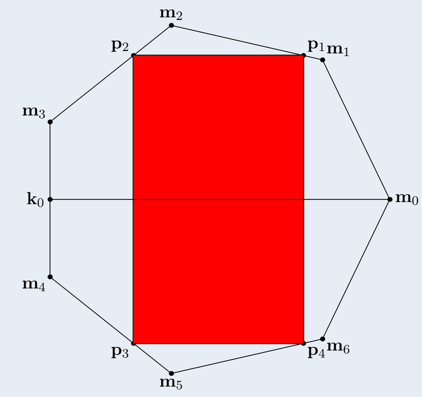
- For covering in $d = 4$ and 5, the optimal lattice is redundantly extreme and so a slightly pointed ball is worse than the ball for covering in a lattice [9].

Dimension $d = 3$: ball locally pessimal

- The 3-ball is a local pessimum for lattice packing among centrally symmetric convex shapes [8].
- Given Kepler's conjecture, the 3-ball is also a local pessimum for general packing among centrally symmetric convex shapes.
- The 3-ball is a local pessimum for lattice covering among centrally symmetric convex shapes [9].

The regular heptagon

- The regular heptagon is a local pessimum with respect to "double lattice"-packing [10].
- If the optimal packing of the regular heptagon is the "double lattice" packing, then it is also a local pessimum for general packing.



References

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[8] Y. Kallus (2012), arXiv:1212.2551.
[9] Y. Kallus (2013), arXiv:1301.5895.
[10] Y. Kallus (2013), arXiv:1305.0289.