# Local optimality/pessimality results in packing

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Applied Math Seminar
Courant Institute, NYU
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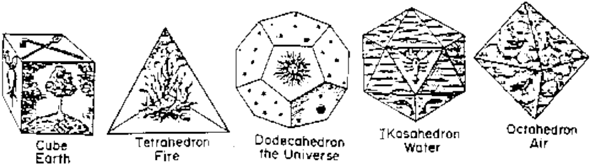
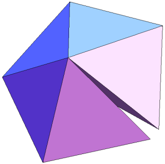
# The long history of packing problems

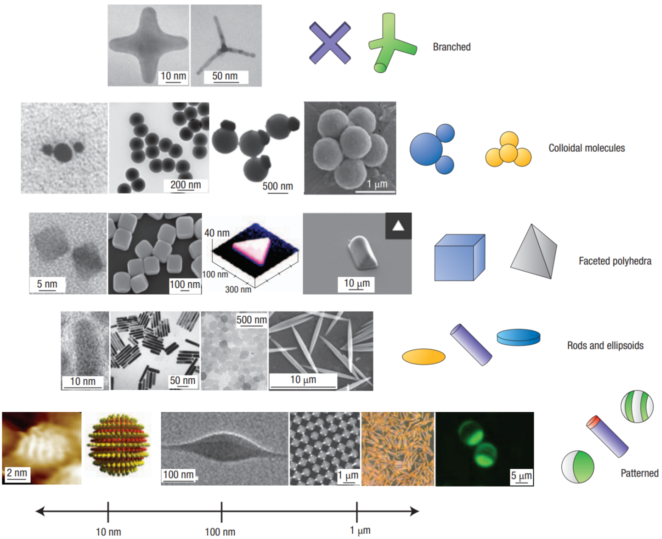


"In general, the attempt to give a shape to each of the simple bodies is unsound, for the reason, first, that they will not succeed in filling the whole. It is agreed that there are only three plane figures which can fill a space, the triangle, the square, and the hexagon, and only two solids, the pyramid [tetrahedron] and the cube."

– Aristotle. *On the Heavens*, volume III

# Building blocks by design




*Glotzer and Solomon, Nature Materials 2007*

# Packing problems in the modern era

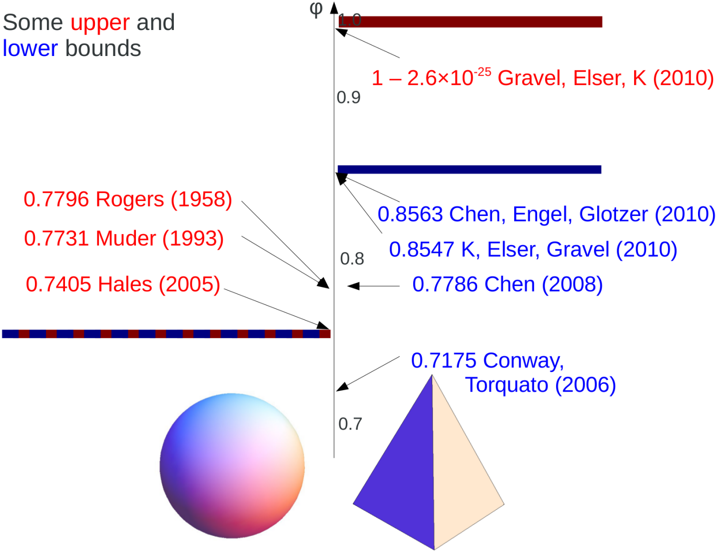
"How can one arrange most densely in space an infinite number of equal solids of a given form, e.g., **spheres** with given radii or regular **tetrahedra** with given edges, that is, how can one so fit them together that the ratio of the filled to the unfilled space may be as large as possible?"

**Theorem (Hales)**

*No sphere packing fills more than* 0.7404 *of space.*

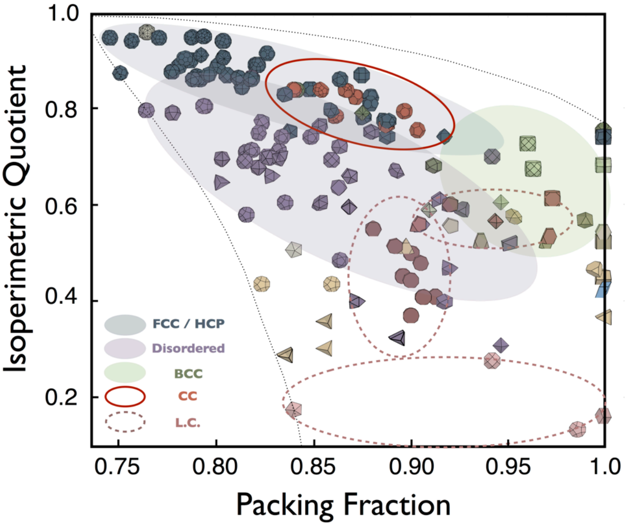
Figures for which optimal packing density is known: space filling tiles, 2D 2-fold-symmetric shapes, 3D spheres (and corollaries).

# Packing regular tetrahedra (2010)



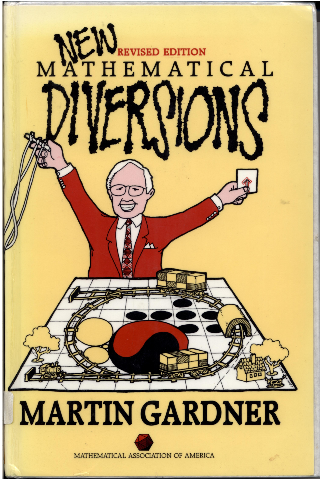

*CIMS Geometry Seminar, February 2010*

# Packing convex shapes




*Damasceno, Engel, and Glotzer, 2012.*

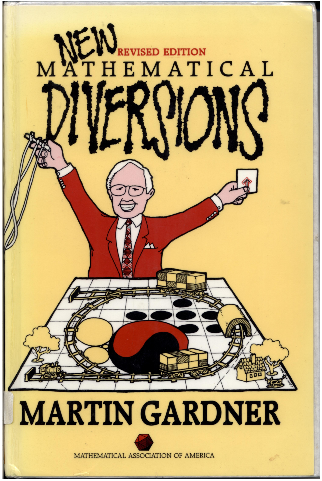
# Ulam's Conjecture




"Stanislaw Ulam told me in 1972 that he suspected the sphere was the worst case of dense packing of identical convex solids, but that this would be difficult to prove."

*1995 postscript to the column "Packing Spheres"*

# Ulam's Last Conjecture

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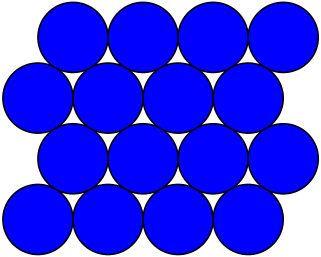
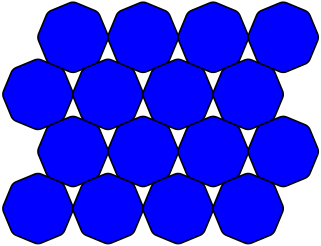
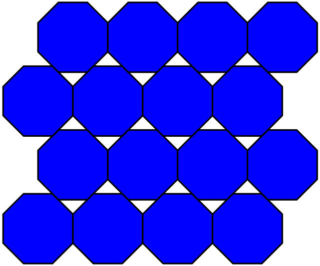
# In 2D disks are not worst



$\phi = 0.9069$

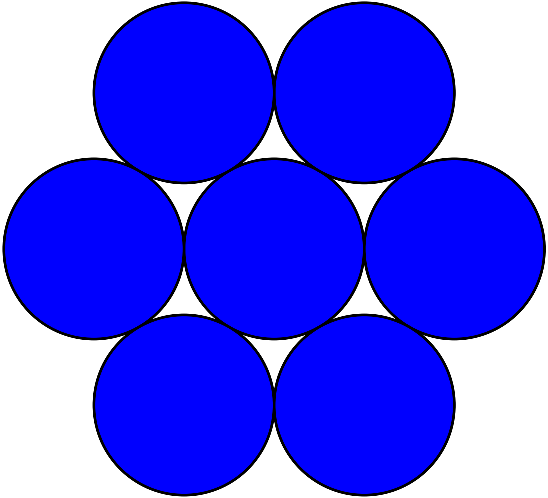


$\phi = 0.9024$



$\phi = 0.9062$

# Why can we improve over circles?



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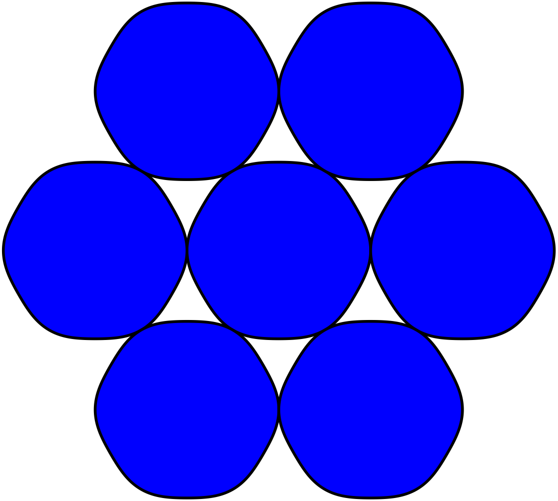

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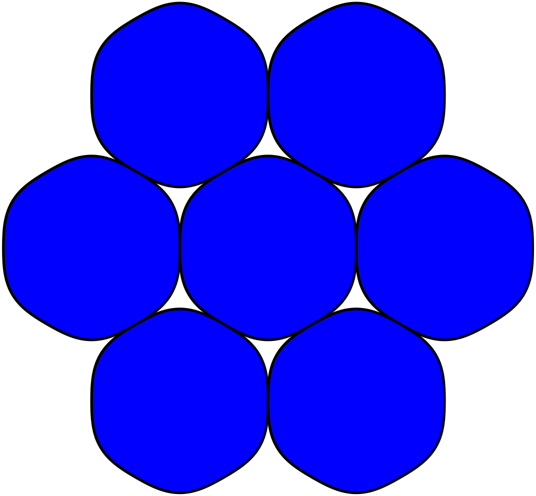

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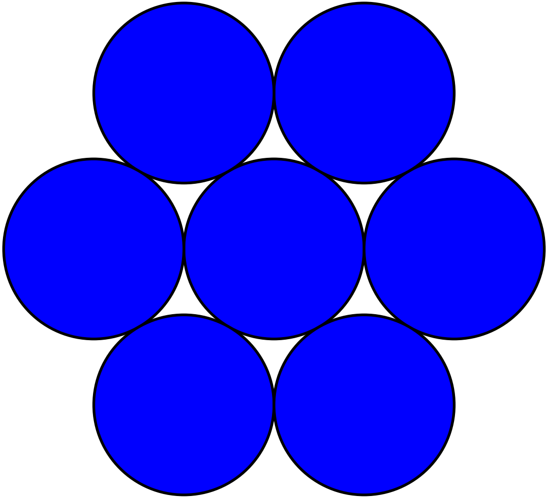

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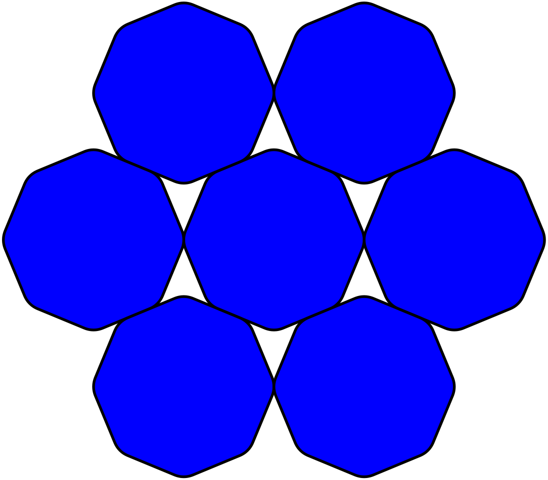

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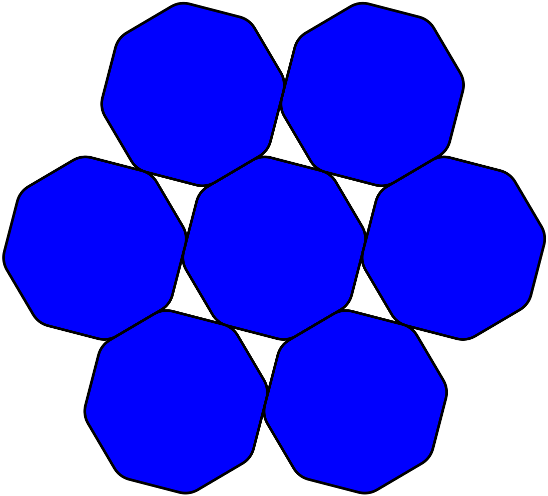

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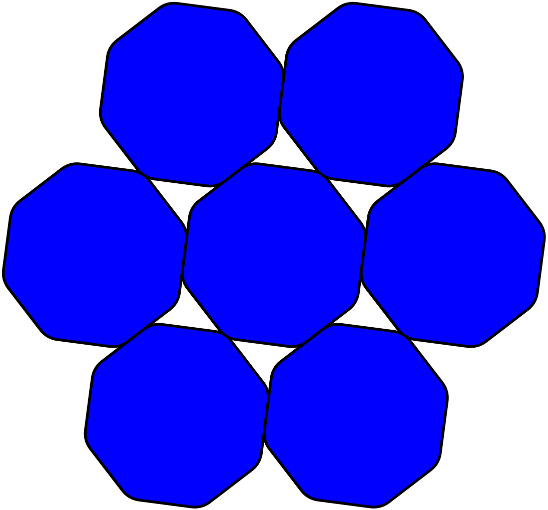

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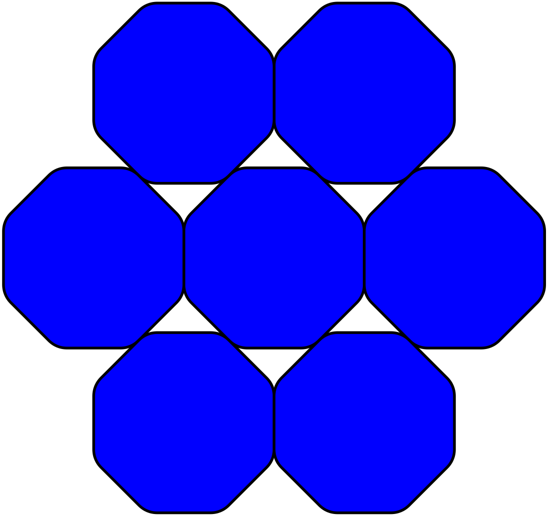

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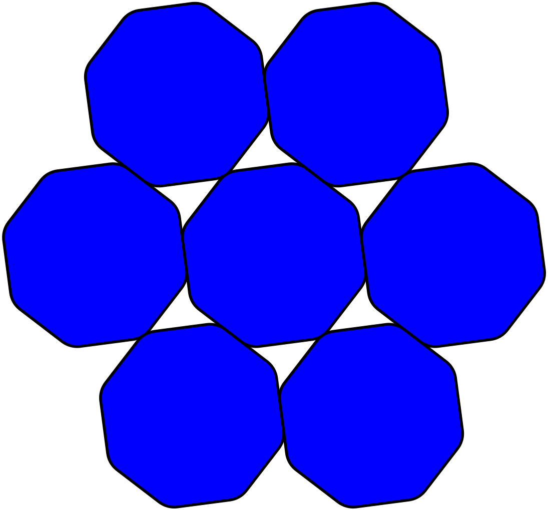

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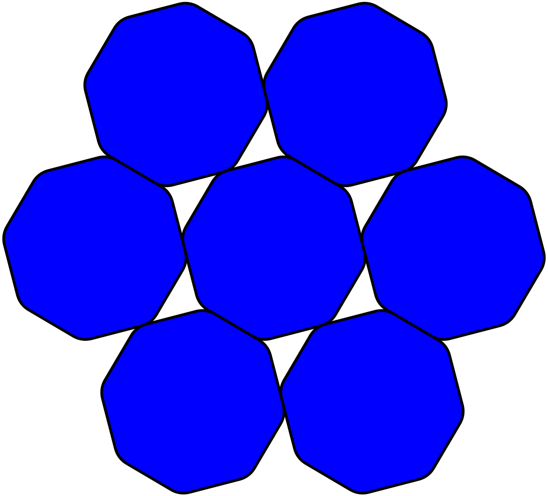

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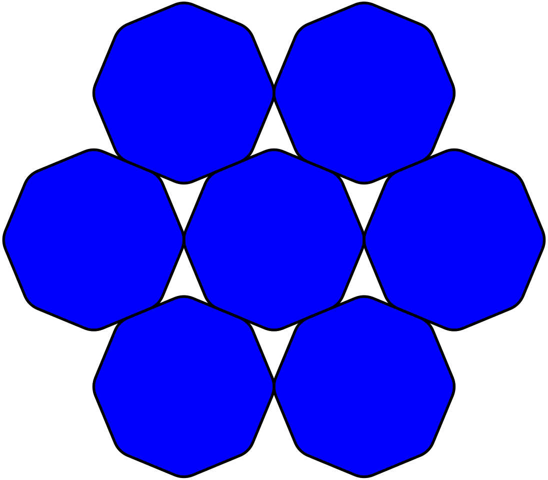

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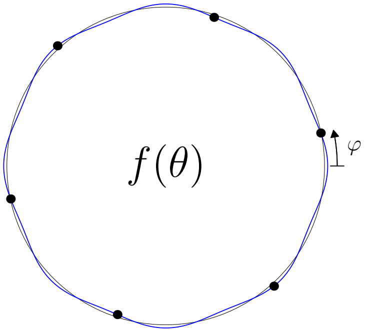


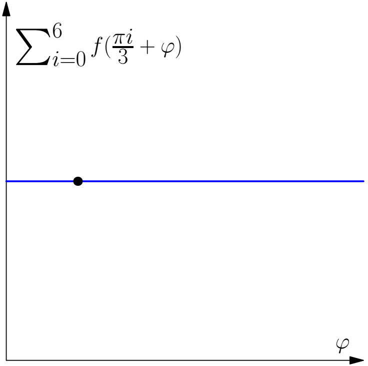

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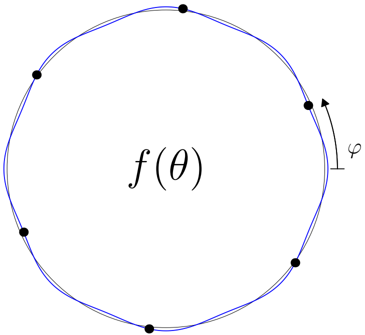


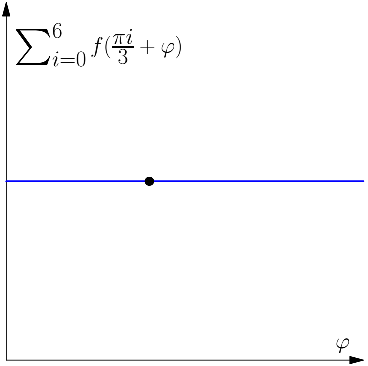

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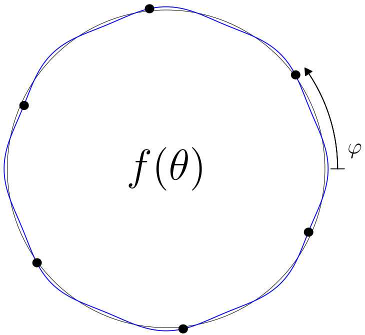


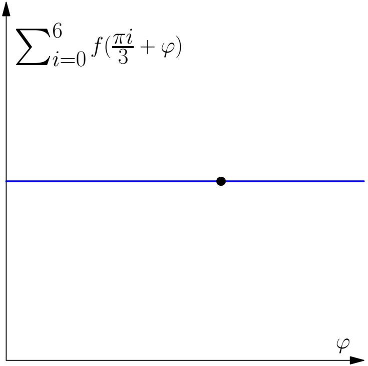

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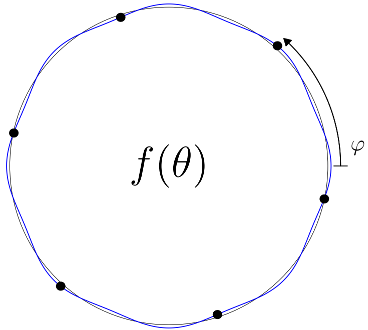


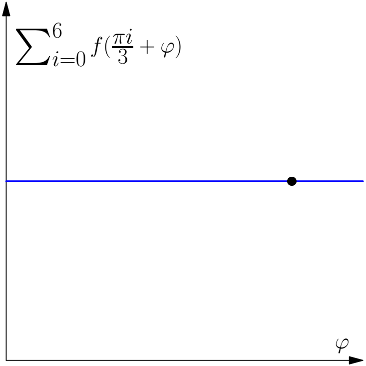

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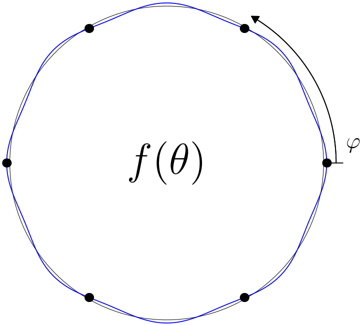


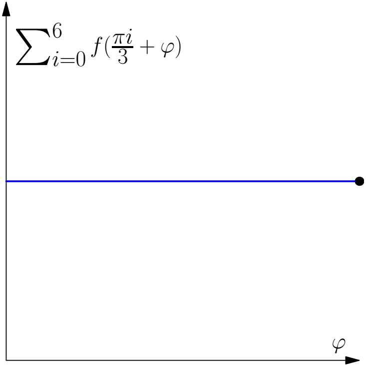

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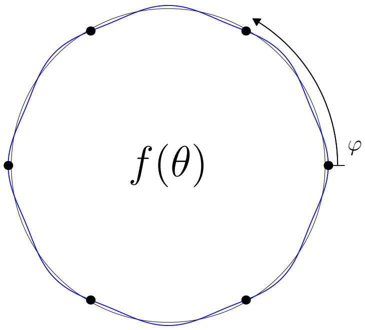
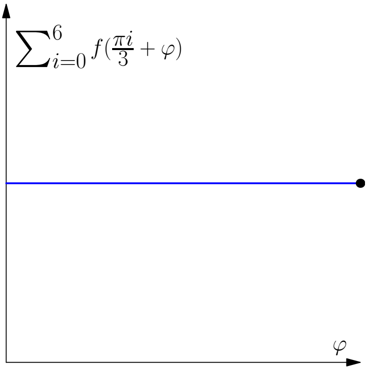
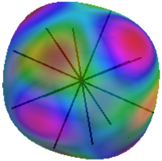
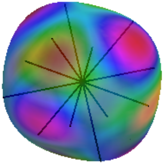


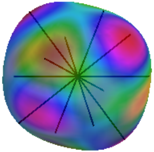




$$f(\theta) = 1 + \epsilon cos(8\theta)$$

# Why can we not improve over spheres?







Let $\mathbf{x}_i$, $i = 1, \ldots, 12$, be the twelve contact points on the sphere in the f.c.c. packing.

**Lemma**

*Let $f$ be an even function $S^2 \to \mathbb{R}$. $\sum_{i=1}^{12} f(R\mathbf{x}_i)$ is independent of $R \in SO(3)$ if and only if the expansion of $f(\mathbf{x})$ in spherical harmonics terminates at $l = 2$.*

*K, Adv Math 2014*

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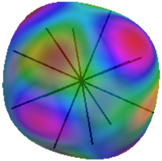
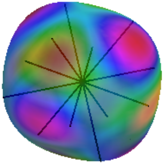
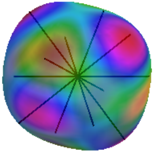






Let $\mathbf{x}_i$, $i = 1, \ldots, 12$, be the twelve contact points on the sphere in the f.c.c. packing.

### Lemma

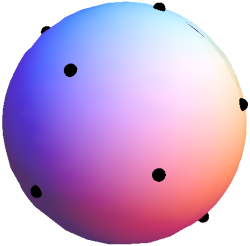
*Let $f$ be an even function $S^2 \to \mathbb{R}$. $\sum_{i=1}^{12} f(R\mathbf{x}_i)$ is independent of $R \in SO(3)$ if and only if the expansion of $f(\mathbf{x})$ in spherical harmonics terminates at $l = 2$.*

### Theorem (K)

*The sphere is a local minimum of $\phi$, the packing density, among convex, centrally symmetric bodies.*

*K, Adv Math 2014*

# Higher dimensions



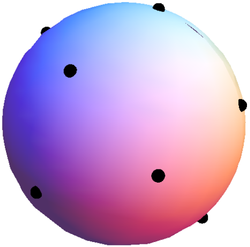
Contact points $S(\Lambda)$ of the optimal lattice.

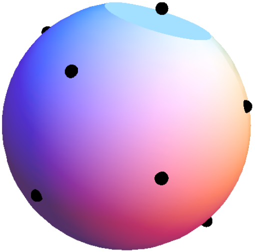
A lattice $\Lambda$ is a image of the integer lattice $\mathbb{Z}^d$ under a linear map $T : \mathbb{R}^d \to \mathbb{R}^d$

A lattice is *admissible* for $K$ if it intersects the interior of $K$ only at the origin. Equivalently, $\{K + 2\mathbf{l} : \mathbf{l} \in \Lambda\}$ is a packing.

*K, Adv Math 2014*

# Higher dimensions





*K, Adv Math 2014*

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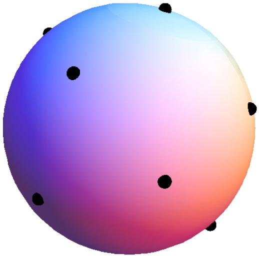
A lattice $\Lambda$ is *extreme* if and only if $||T\mathbf{x}|| \geq ||\mathbf{x}||$ for all $\mathbf{x} \in S(\Lambda) \Longrightarrow \det T > 1$ for $T \approx 1$.

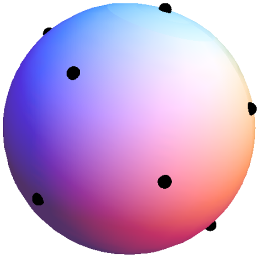
In $d = 6, 7, 8, 24$, the optimal lattice is *redundantly extreme*, and so the ball is *reducible*.

# $d = 4$ and $d = 5$



In $d = 4, 5$, if $||T\mathbf{x}|| \geq ||\mathbf{x}||$ for all $\mathbf{x} \in S(\Lambda) \setminus \{\mathbf{x}_0\}$, and $||T\mathbf{x}_0|| > (1-\epsilon)||\mathbf{x}_0||$, then $1 - \det T < C\epsilon^2$ (compared with $C\epsilon$ for $d = 2, 3$).

# $d = 4$ and $d = 5$



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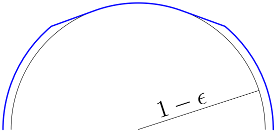
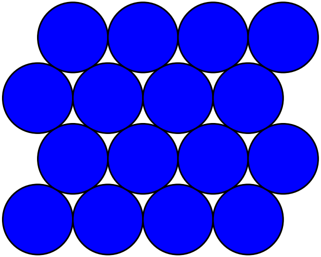



$$(\rho(K) - \rho(B))/\rho(B) \sim \epsilon^2$$
$$(V(B) - V(K))/V(B) \sim \epsilon$$

The ball is not a local minimum of the optimal packing fraction.

# In 2D disks are not worst



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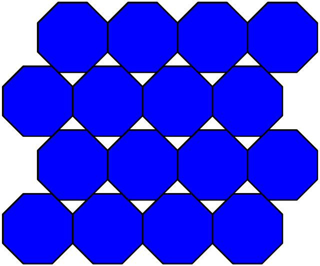

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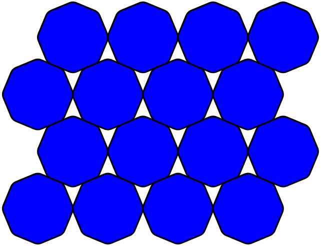
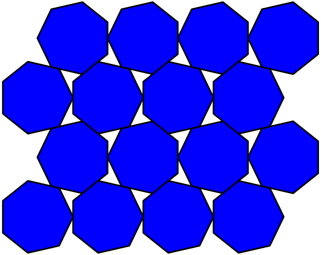
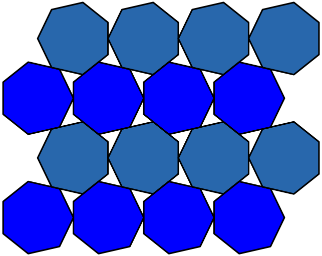


$\phi =$ 0.9024



$\phi =$ 0.8926(?)

# Regular heptagon is locally worst packing (?)



0.8926(?)

**Theorem (K)**

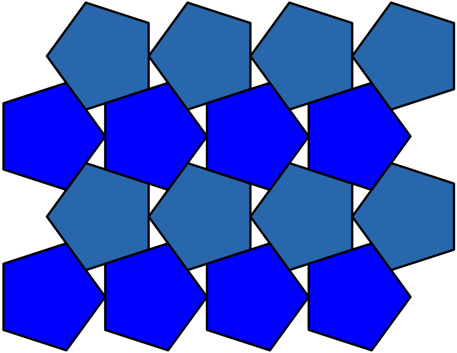
*Any convex body sufficiently close to the regular heptagon can be packed at a filling fraction at least that of the "double lattice" packing of regular heptagons.*

It is not proven, but highly likely, that the "double lattice" packing is the densest packing of regular heptagons.

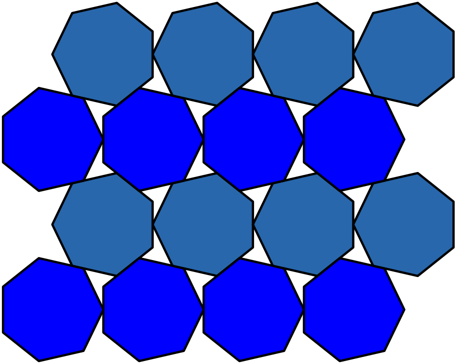
*K, Geometry & Topology 2015*

# Local optimality of the double lattice packing

Work with Wöden Kusner (TU Graz)

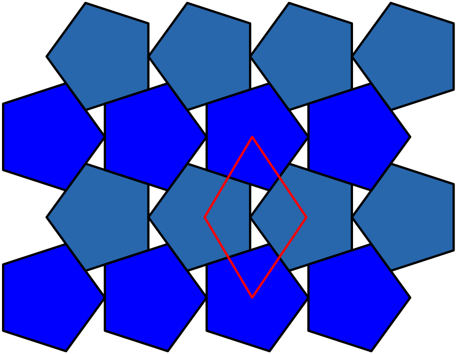


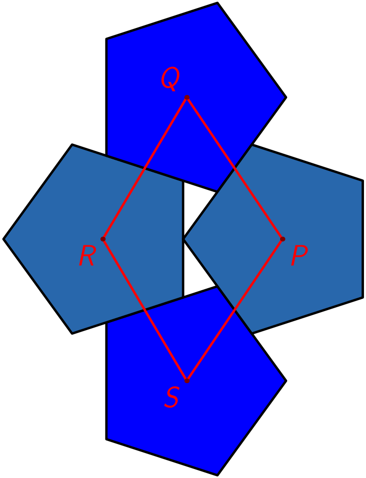
$\phi = 0.9213$



$\phi = 0.8926$

# Pentagons

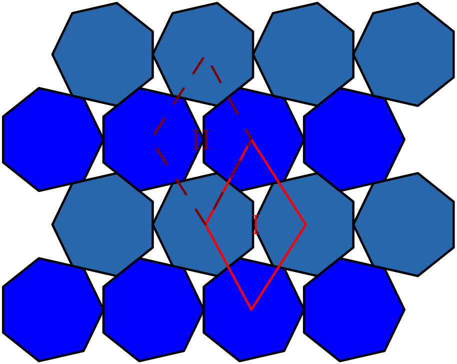


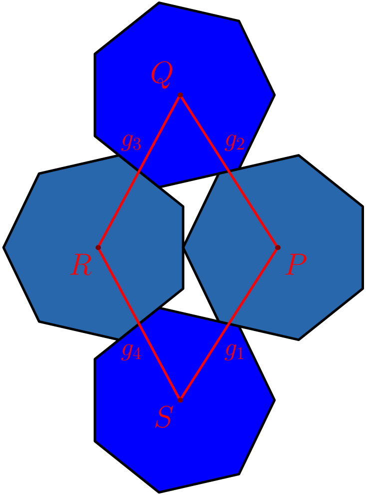



This configuration is a local minimum among nonoverlapping configurations of $\mathrm{area}(SPQR)$.

*K and Kusner, arXiv:1509.02241*

# Heptagons

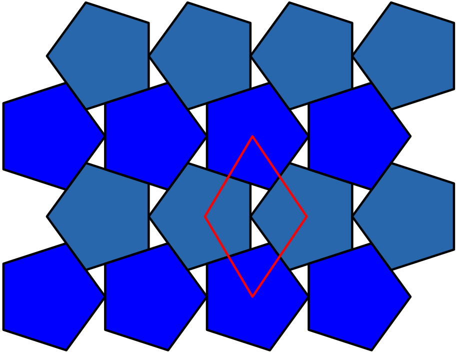


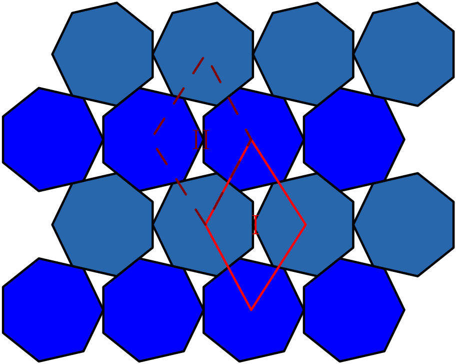

This configuration is not a local minimum of $\mathrm{area}(SPQR)$.
But it is a local minimum of $\mathrm{area}(SPQR) + \sum_{i=1}^{4} g_i$, where, e.g., $g_3^{(I)} + g_3^{(II)} = 0$.

*K and Kusner, arXiv:1509.02241*

# Local optimality of the double lattice packing




$\phi = 0.9213$




$\phi = 0.8926$

The same method that works for heptagons works for (almost) any convex polygon and shows the "double lattice" construction gives locally optimal packings.

*K and Kusner, arXiv:1509.02241*