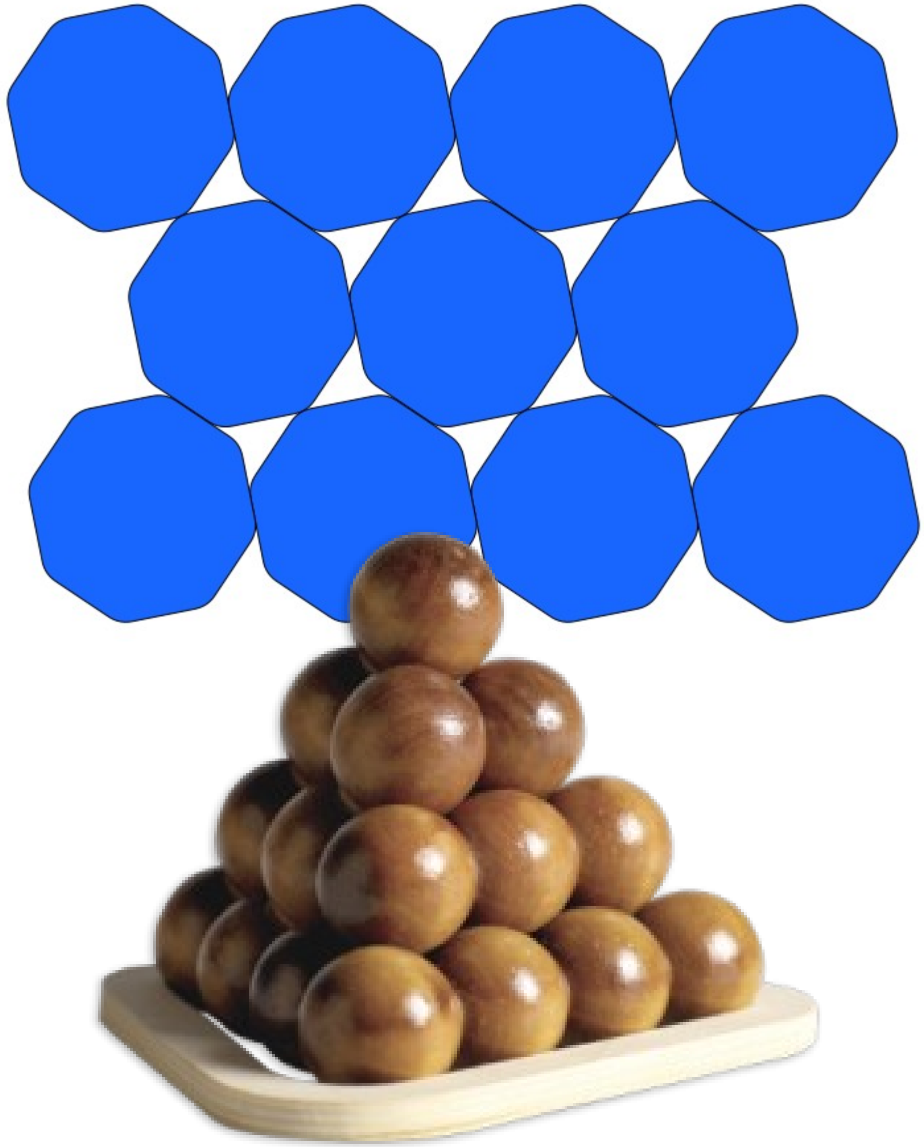
# Worst Packing Shapes



Yoav Kallus
Princeton University
j/w: Fedor Nazarov
Kent University

International Workshop on Packing Problems
Trinity College, Dublin
Sep. 3, 2012

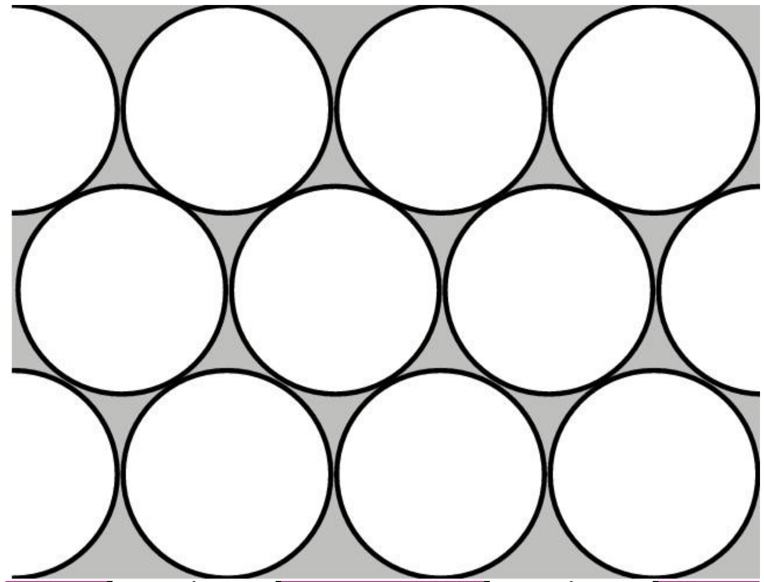
# The Miser's Problem

A miser is required by a contract to deliver a tray covered with a single layer of gold coins, filling the tray as densely as possible. The coins must be convex, of a certain width, and much smaller than the tray. What shape of coin should the miser choose so as to part with as little gold as possible?

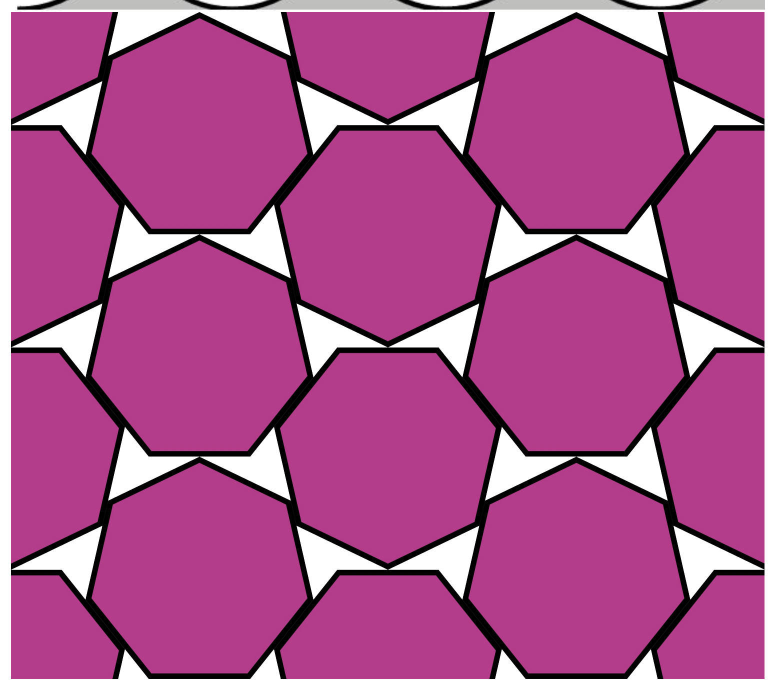
# The Miser's Problem

A miser is required by a contract to deliver a tray covered with a single layer of gold coins, filling the tray as densely as possible. The coins must be convex, of a certain width, and much smaller than the tray. What shape of coin should the miser choose so as to part with as little gold as possible?

# The Miser's Problem

0.9069

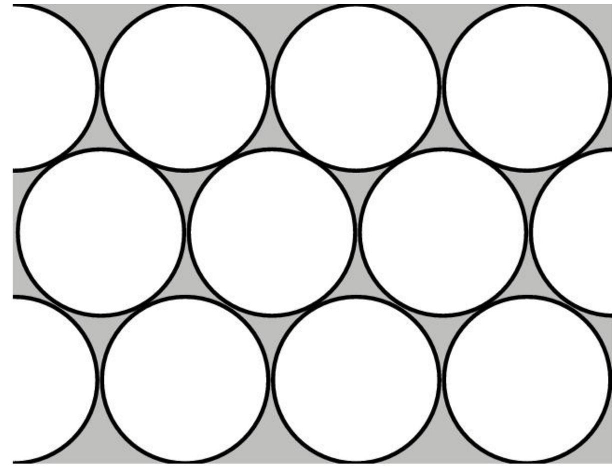



0.8926

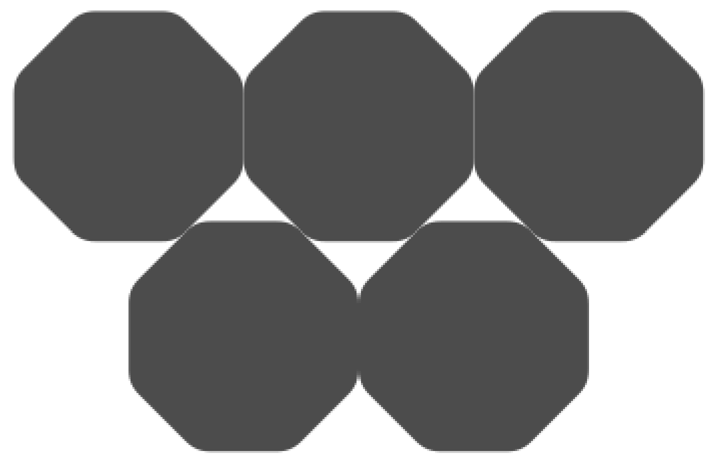
# The Miser's Problem

A miser is required by a contract to deliver a tray covered with a single layer of gold coins, arranged in a **lattice** as densely as possible. The coins must be convex, **centrally symmetric**, of a certain width, and much smaller than the tray. What shape of coin should the miser choose so as to part with as little gold as possible?

# The Miser's Problem

0.9069

0.9024

Conjecture (Reinhardt 1934): rounded octagon is the worst symmetric packer.

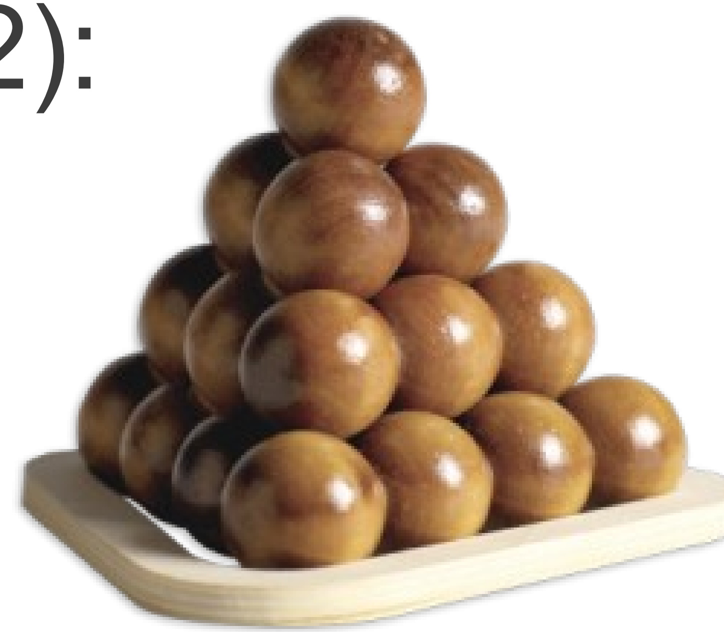
# The Miser's 3D Problem

A miser is required by a contract to deliver a chest of gold bars, arrange in a lattice as densely as possible. The bars must be convex, centrally symmetric, and much smaller than the chest. What shape of bar should the miser cast so as to part with as little gold as possible?

# The Miser's 3D Problem

A miser is required by a contract to deliver a chest of gold bars, arrange in a lattice as densely as possible. The bars must be convex, centrally symmetric, and much smaller than the chest. What shape of bar should the miser cast so as to part with as little gold as possible?

Conjecture (Ulam 1972): ball is the worst symmetric packer.



# Some terminology

$\delta_L(K)$ = optimal number density

$\phi_L(K)$ = optimal packing fraction

= $\delta_L(K)V(K)$

Note: $\phi_L(K) = \phi_L(TK)$

$\phi_L(K)$ is defined over the Banach-Mazur Compactum.

# Some terminology

Note: $\delta_L(K) \leq \delta_L(K')$ if $K' \subset K$

Definition: if the inequality is strict whenever K' is strictly contained in K, then K is "irreducible".

Note: if K is a local minimum of $\phi_L(K)$ then K is irreducible.

# Some terminology

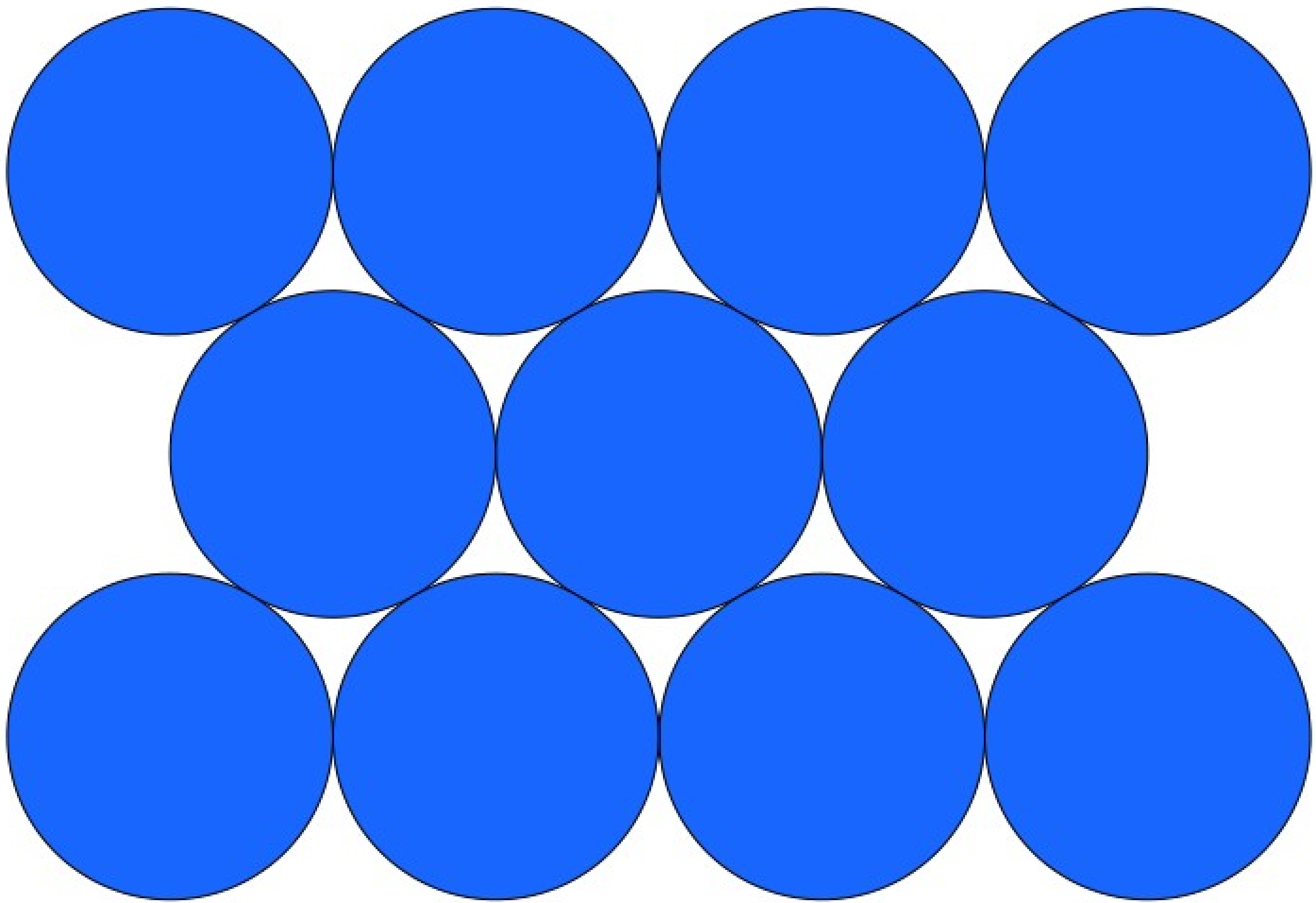
Note: $\delta_L(K) \leq \delta_L(K')$ if $K' \subset K$

Definition: if the inequality is strict whenever K' is strictly contained in K, then K is “irreducible”.

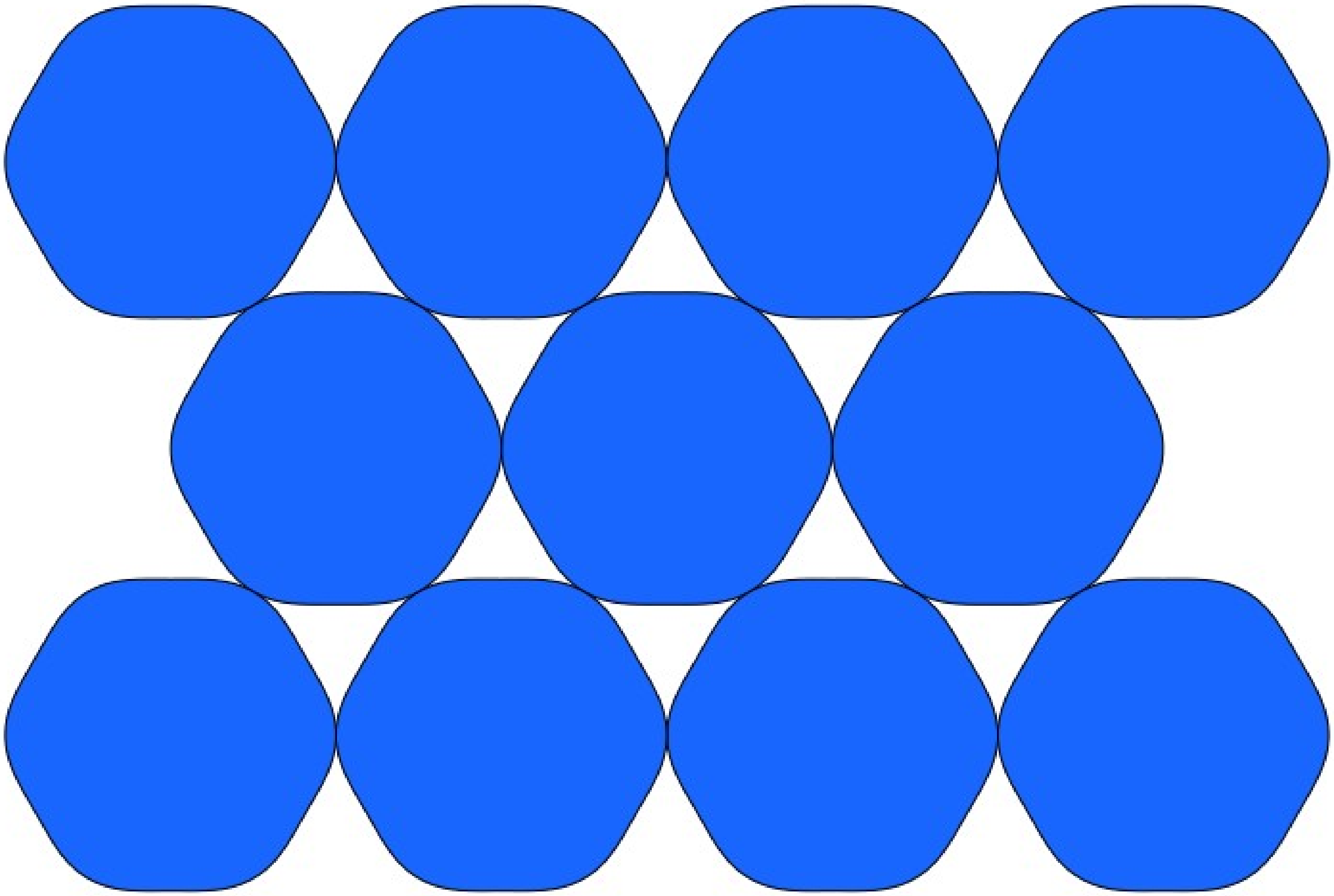
Note: if K is a local minimum of $\phi_L(K)$ then K is irreducible.

- The rounded octagon is a local minimum of optimal packing fraction (Nazarov 1986).
- The circle is irreducible, but is not a local minimum.

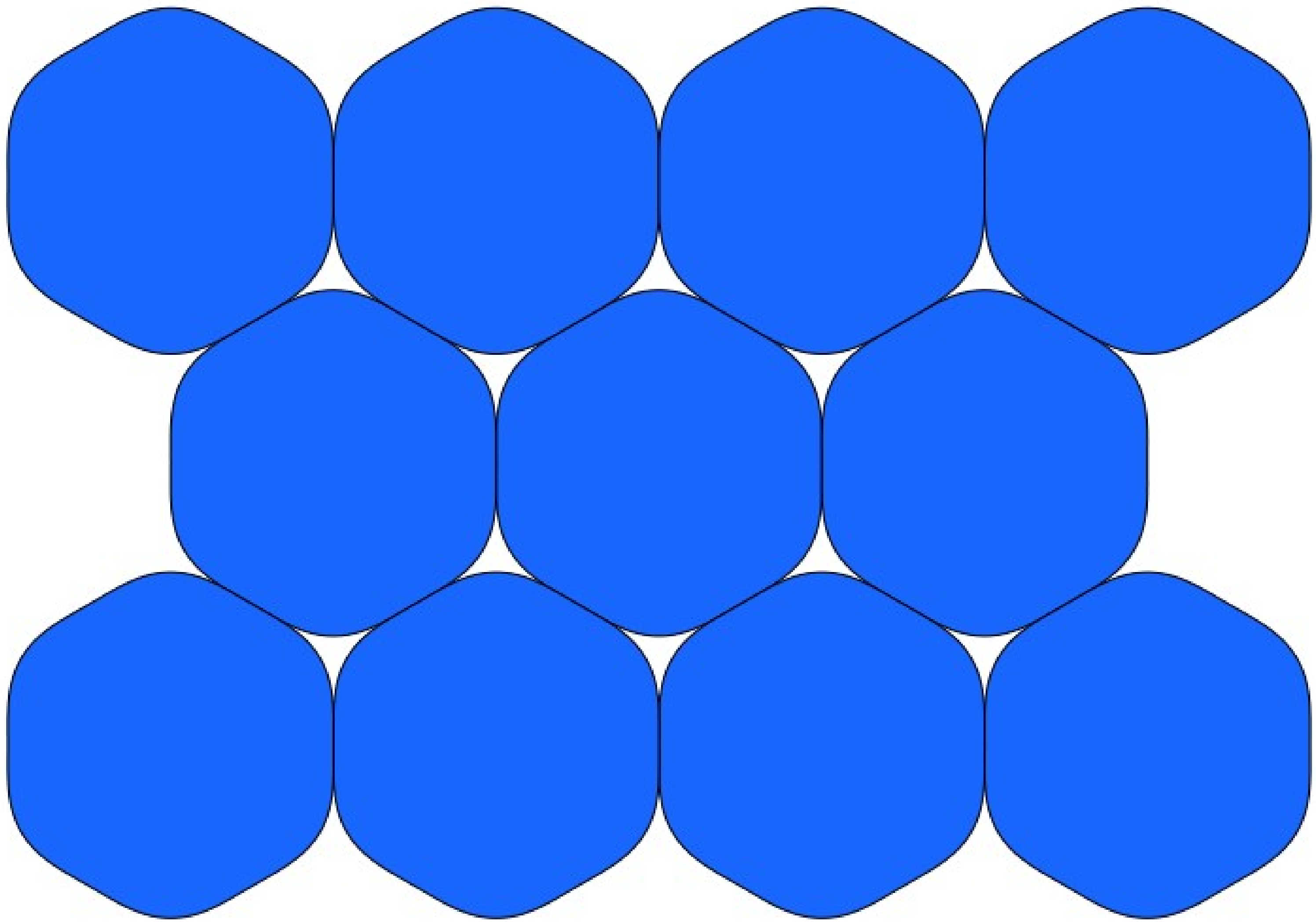
# Why can we improve over circles?



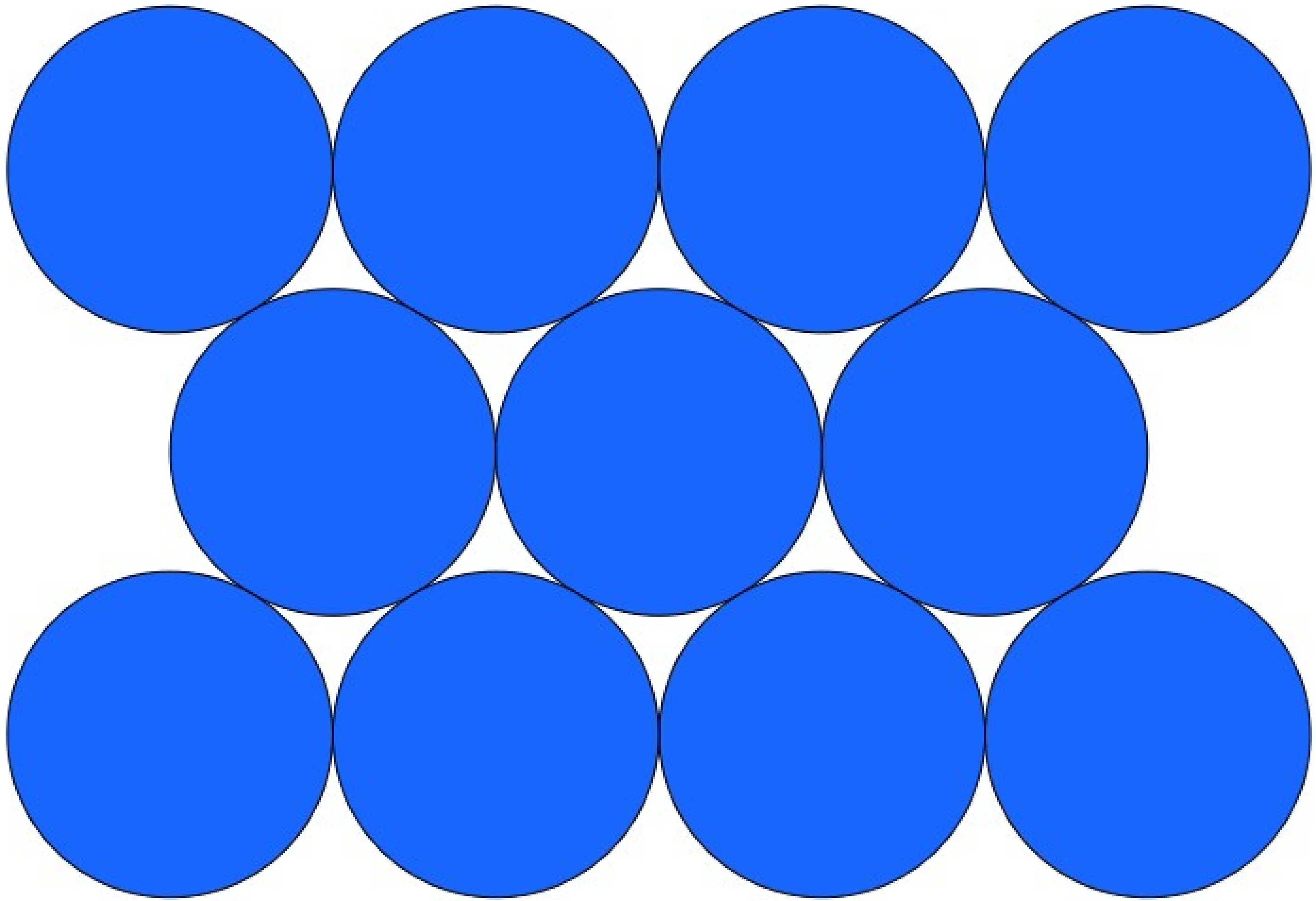
# Why can we improve over circles?



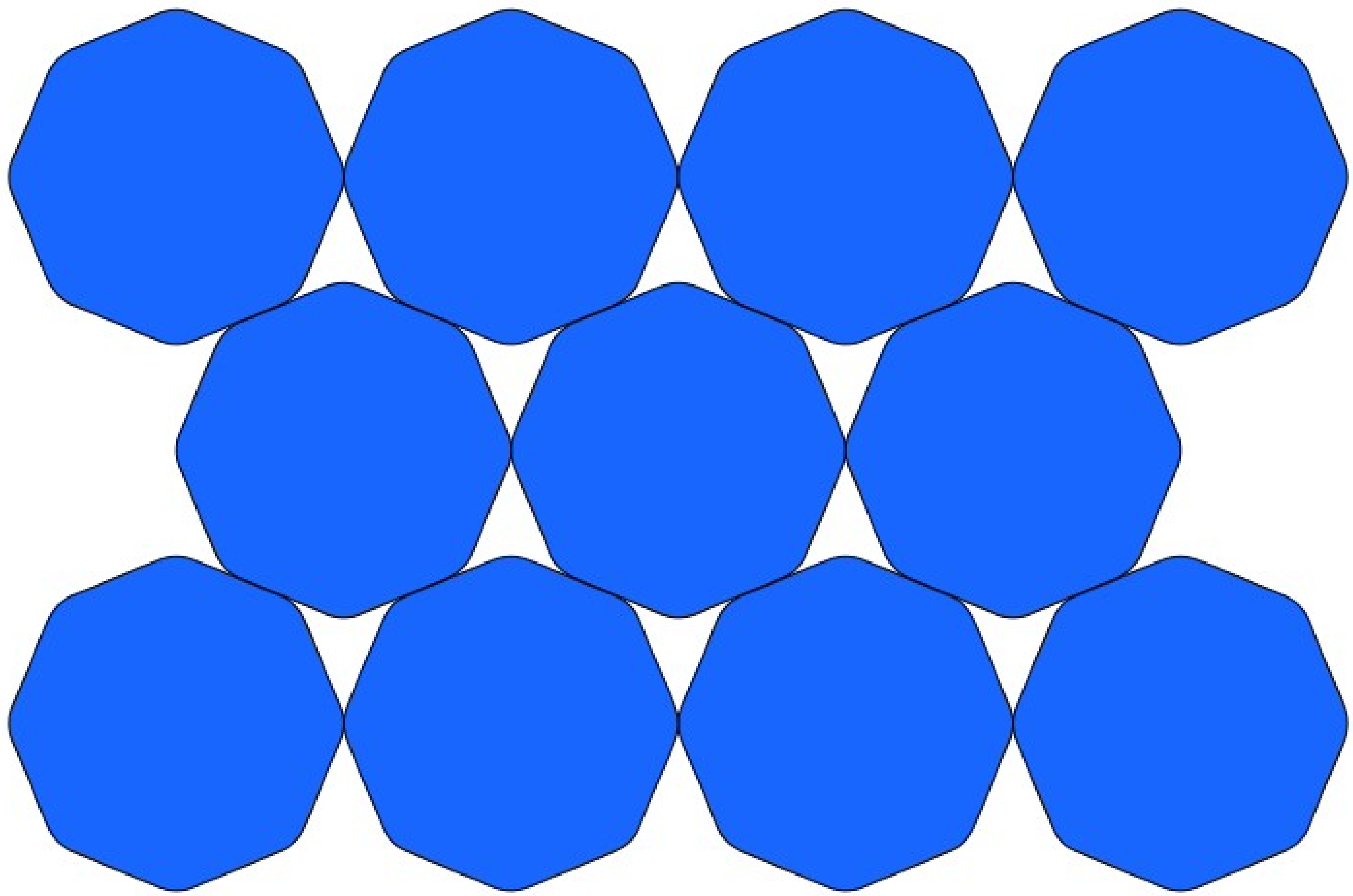
# Why can we improve over circles?



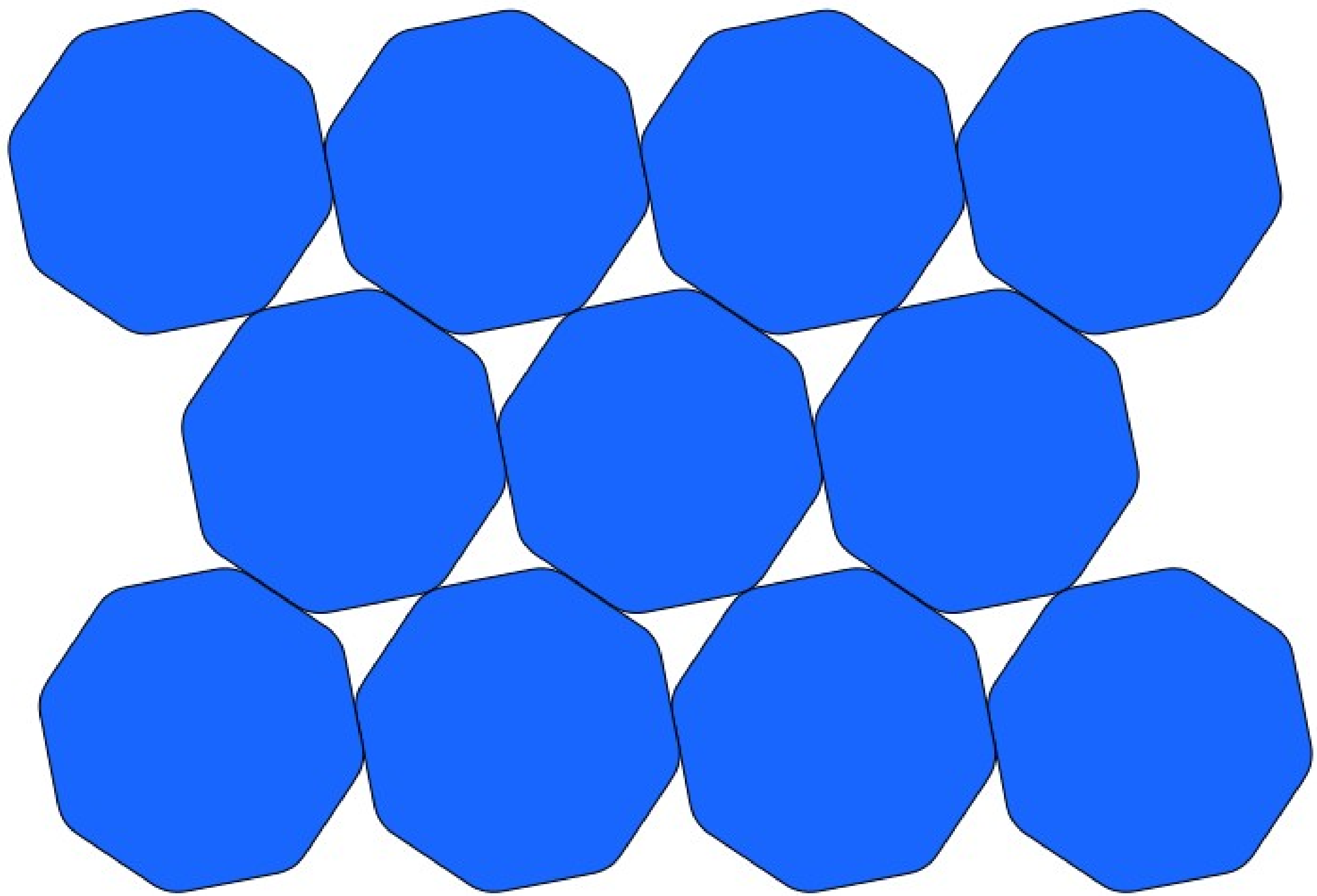
# Why can we improve over circles?



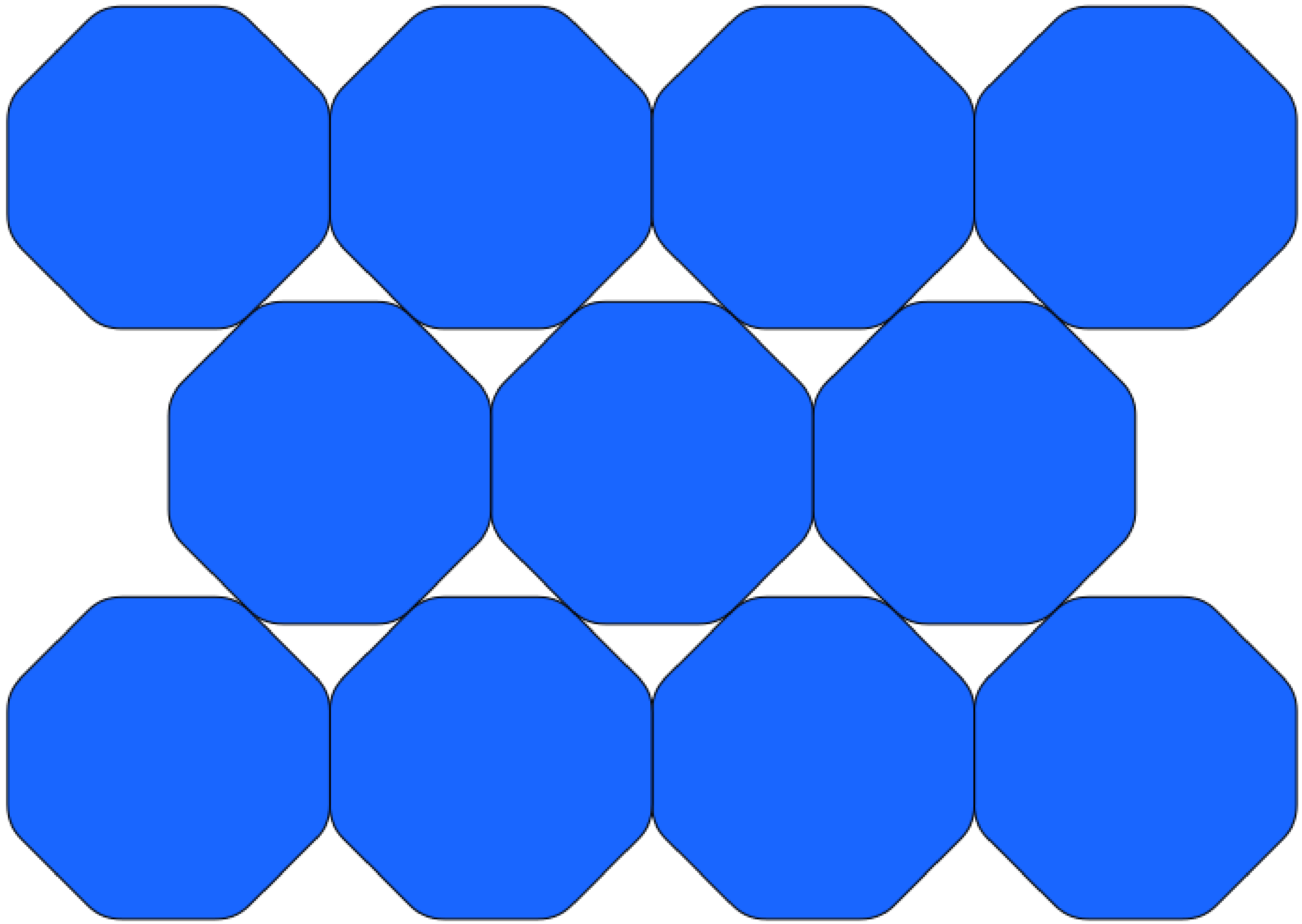
# Why can we improve over circles?



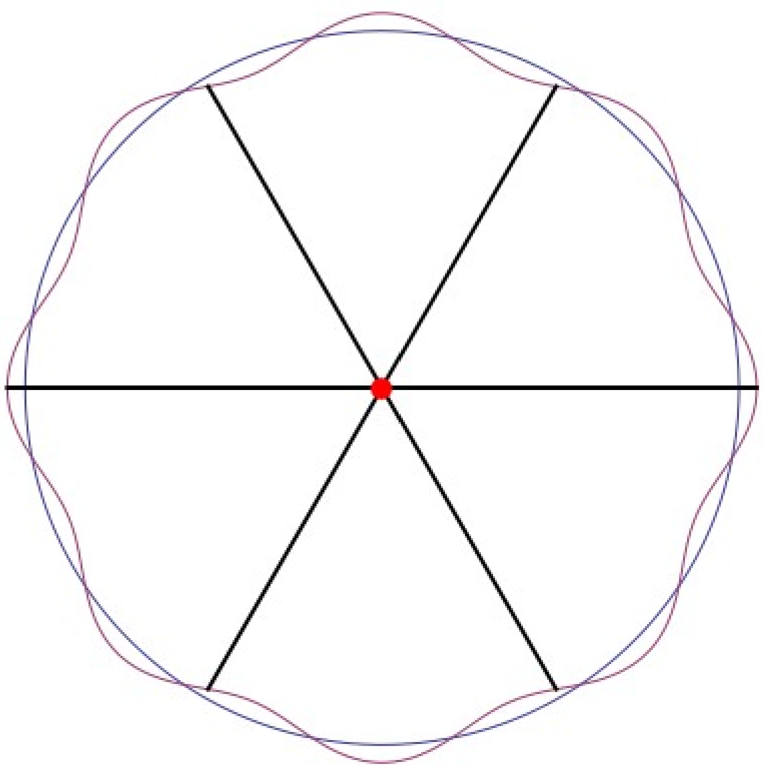
# Why can we improve over circles?

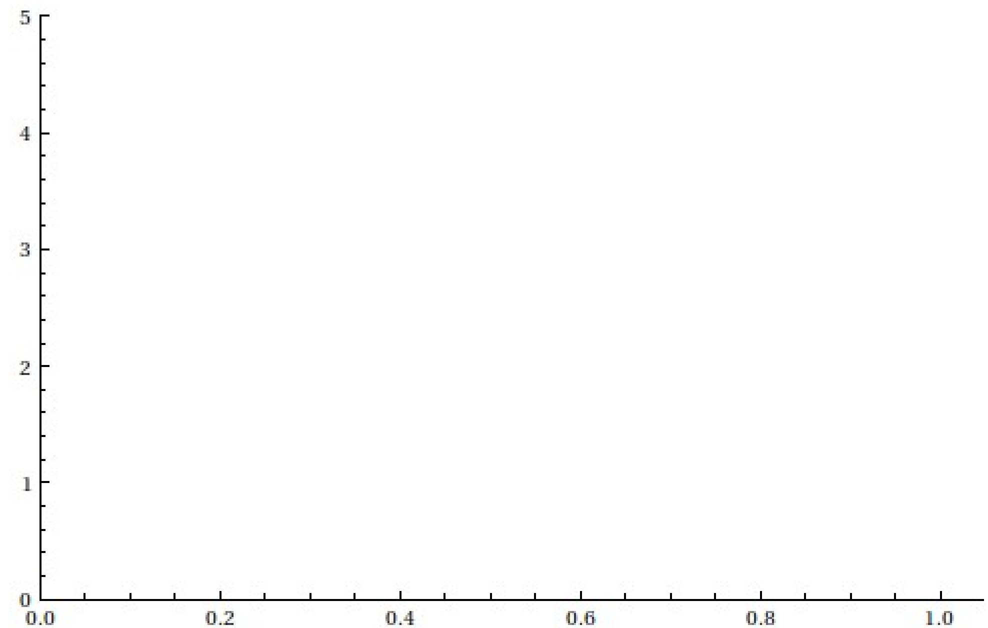


# Why can we improve over circles?



# Why can we improve over circles?

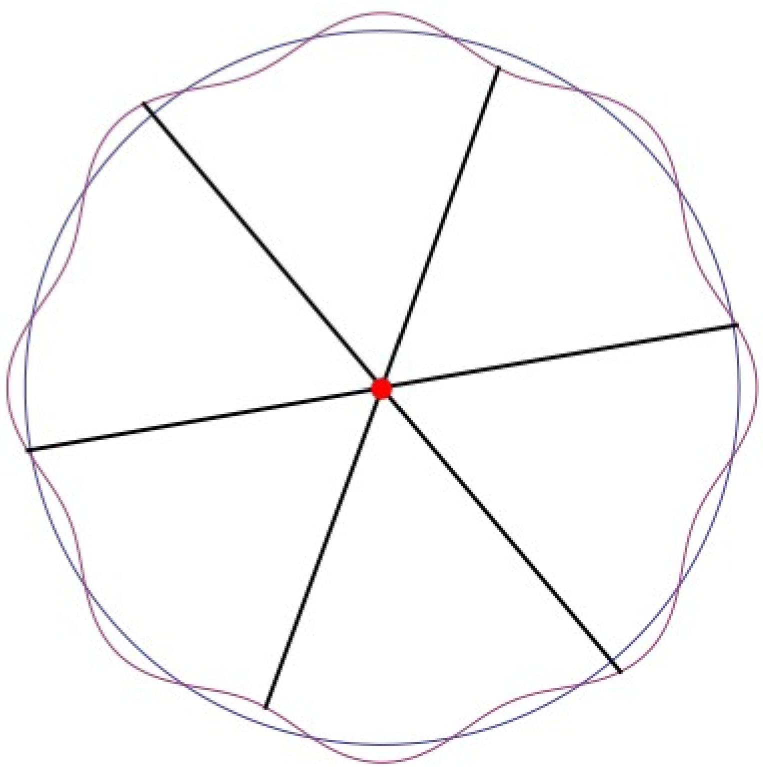


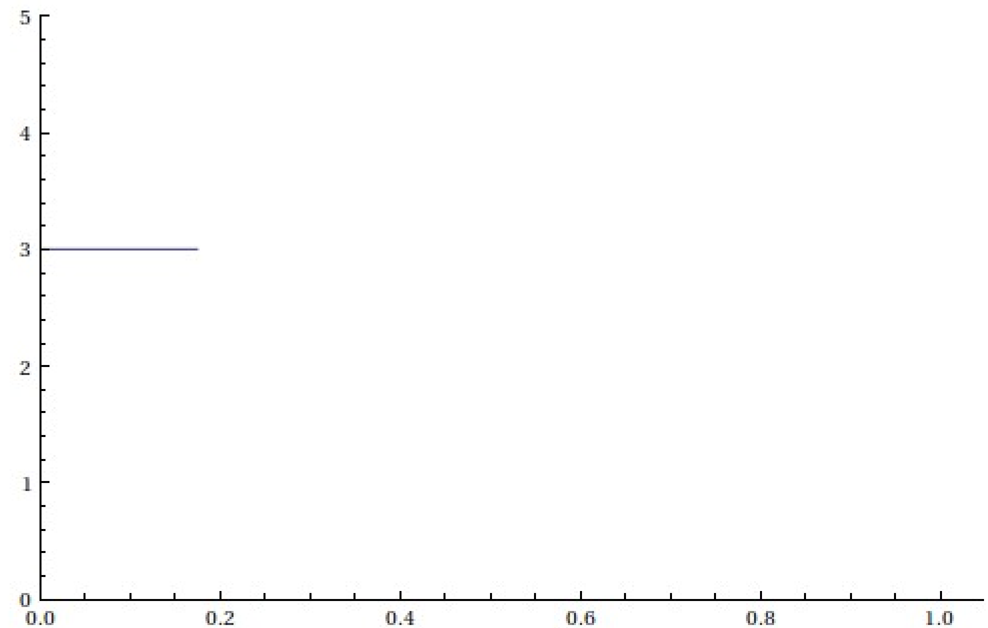



$$f(\theta)$$

$$\sum_{i=1}^{6} f(\theta_i + \phi)$$

# Why can we improve over circles?

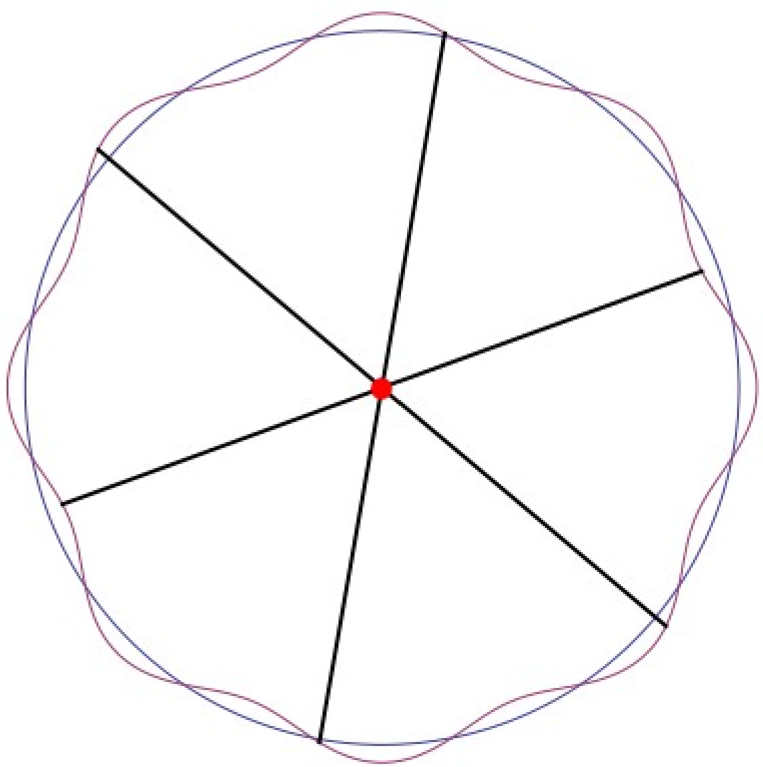


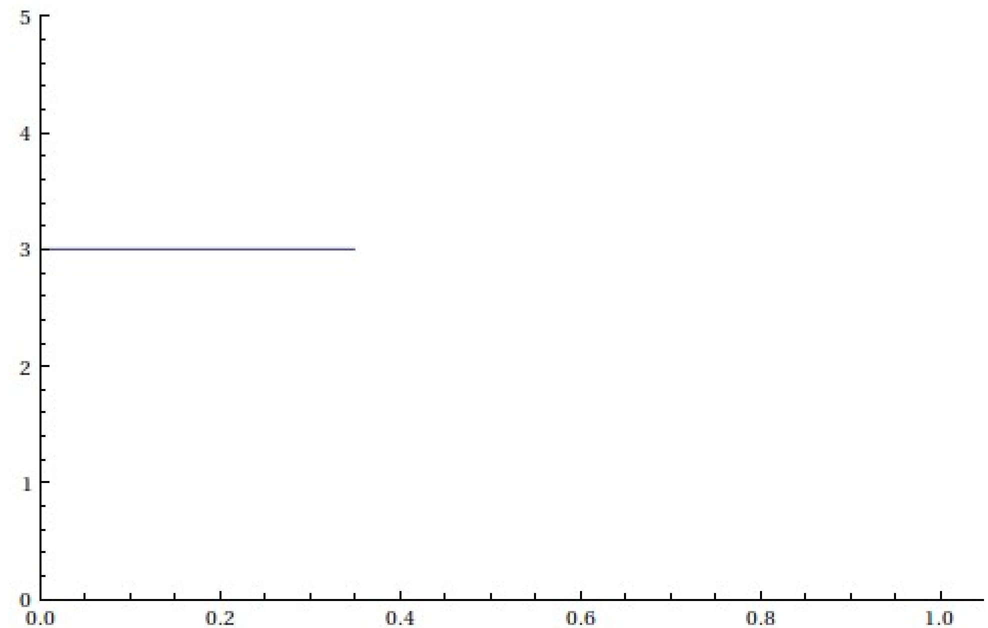



$$f(\theta)$$

$$\sum_{i=1} f(\theta_i + \phi)$$

# Why can we improve over circles?

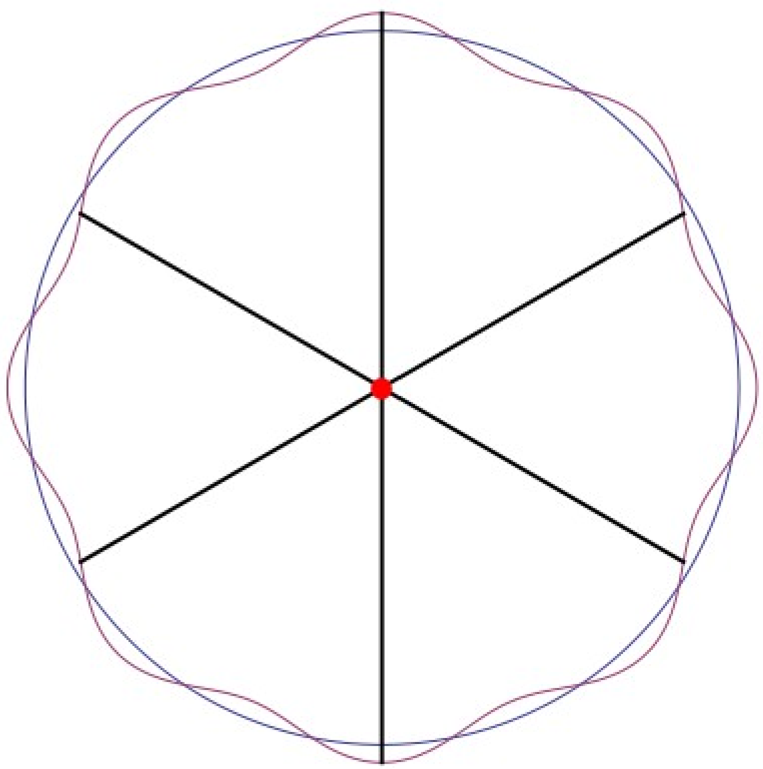


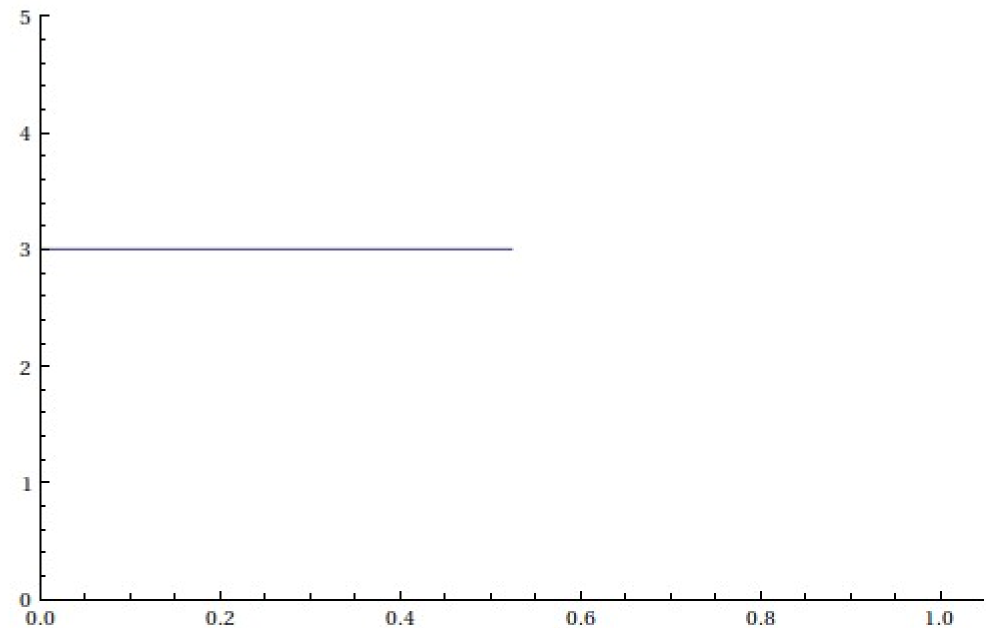



$$f(\theta)$$

$$\sum_{i=1} f(\theta_i + \phi)$$

# Why can we improve over circles?

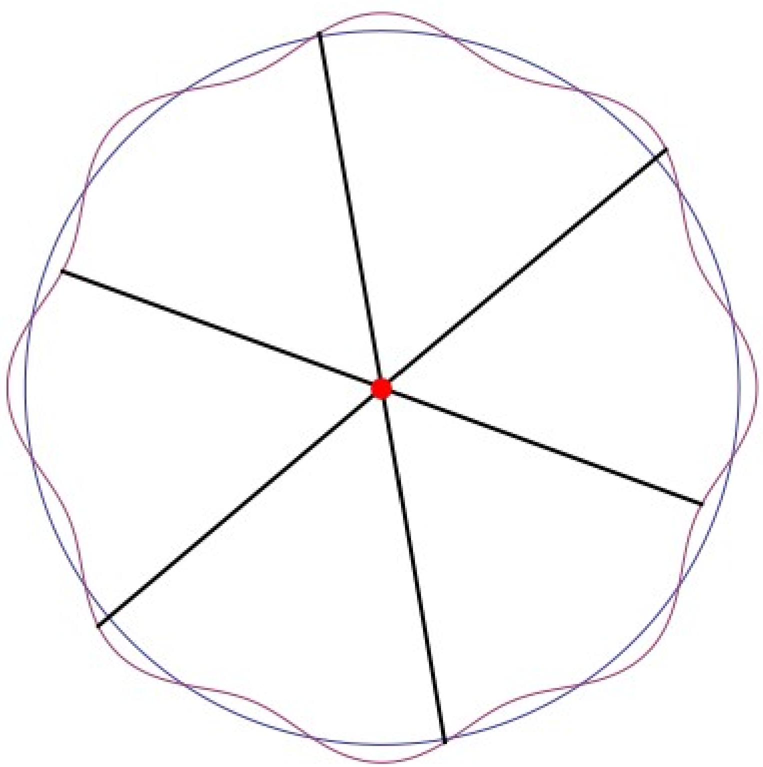


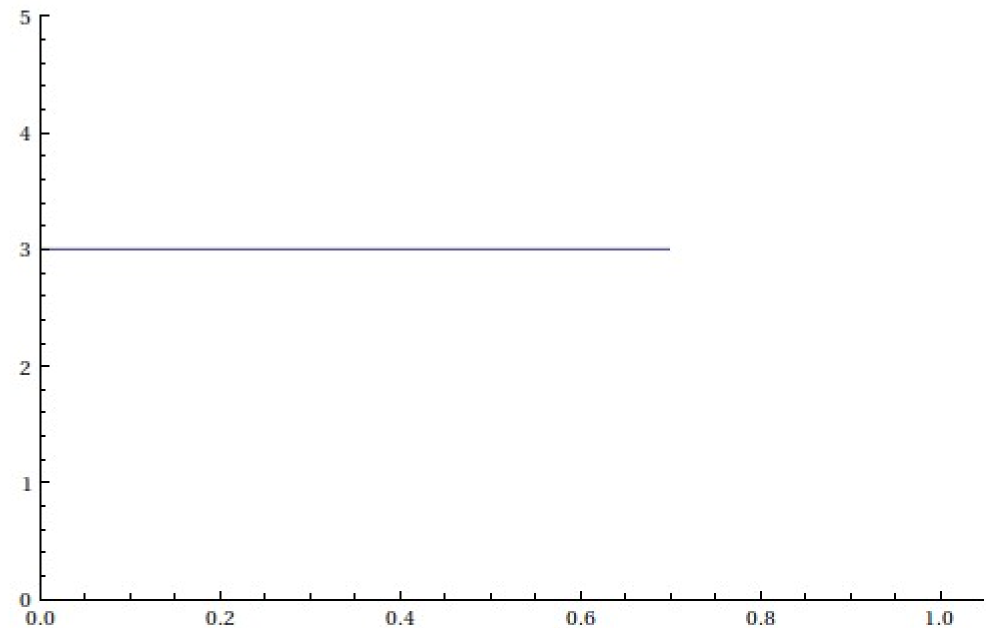



$$f(\theta)$$

$$\sum_{i=1} f(\theta_i + \phi)$$

# Why can we improve over circles?

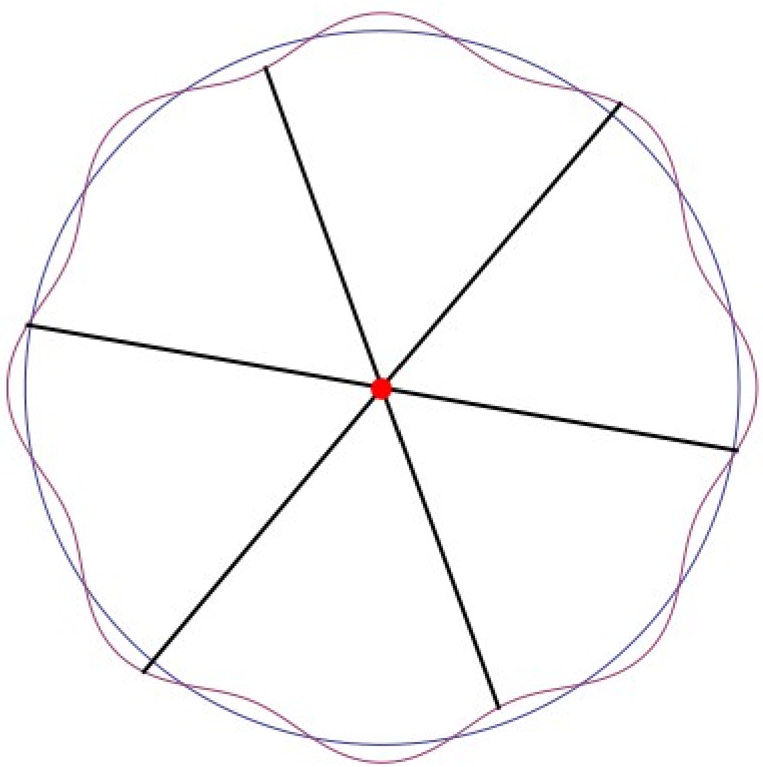


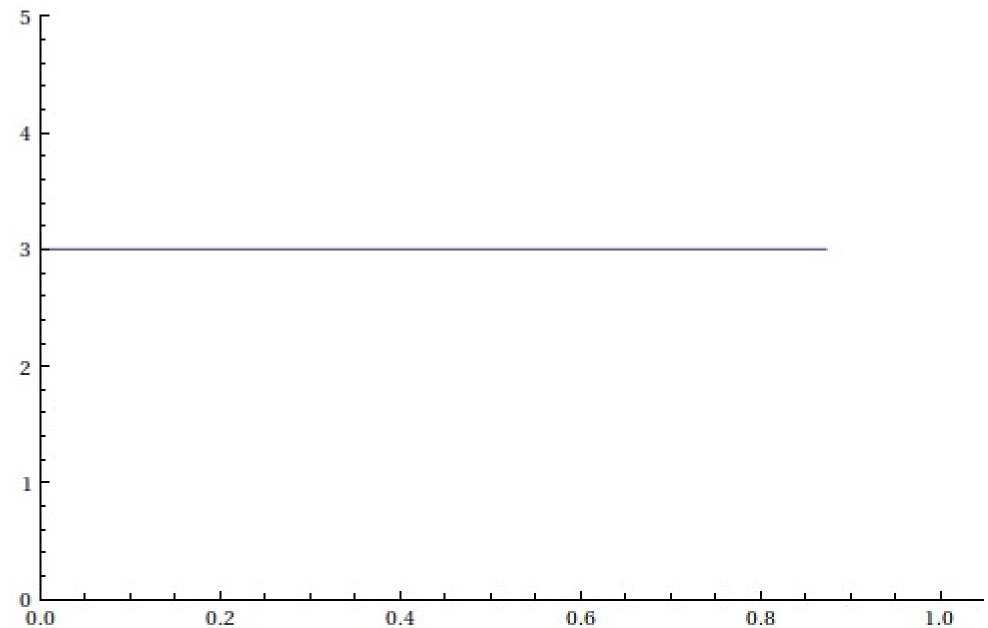



$$f(\theta)$$

$$\sum_{i=1} f(\theta_i + \phi)$$

# Why can we improve over circles?

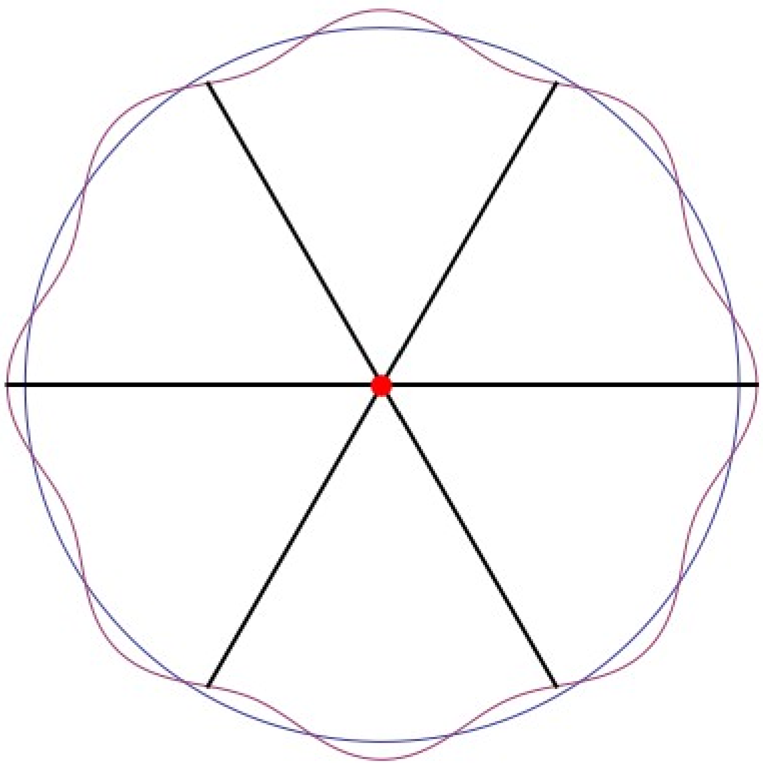


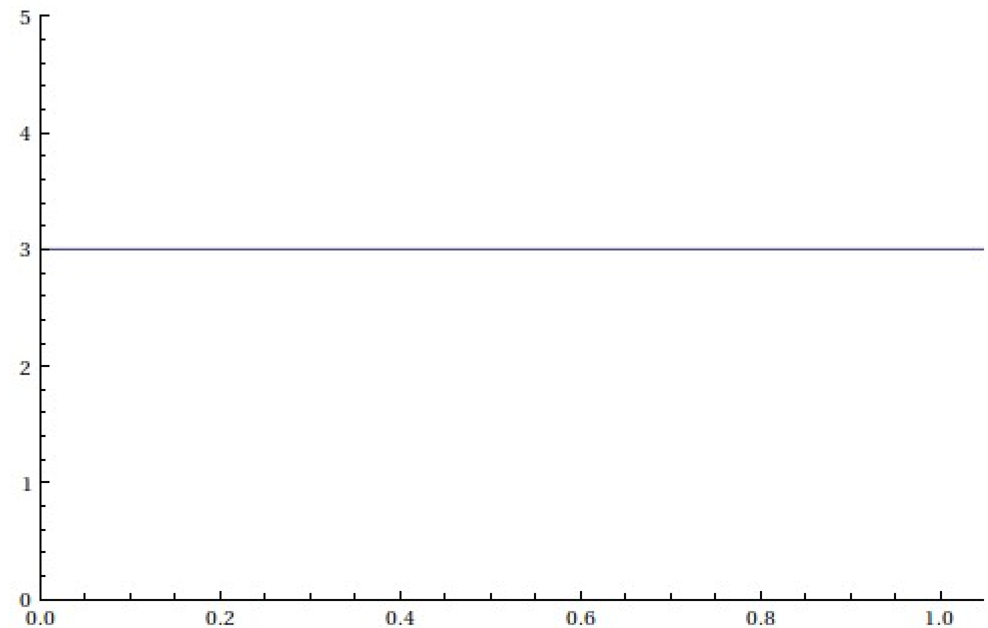



$$f(\theta)$$

$$\sum_{i=1} f(\theta_i + \phi)$$

# Why can we improve over circles?

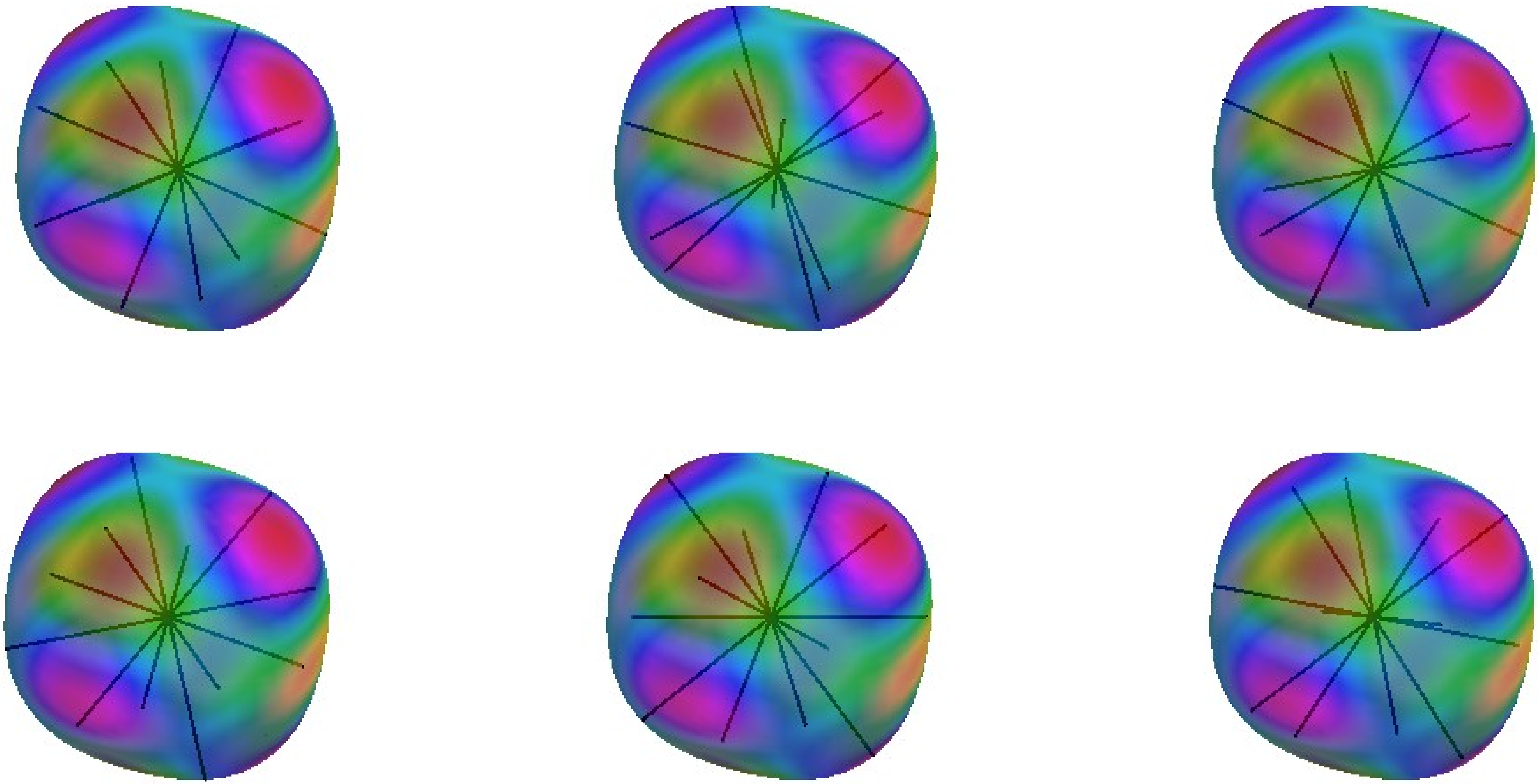





$$f(\theta)$$

$$\sum_{i=1} f(\theta_i + \phi)$$

# Why can we not improve over spheres?



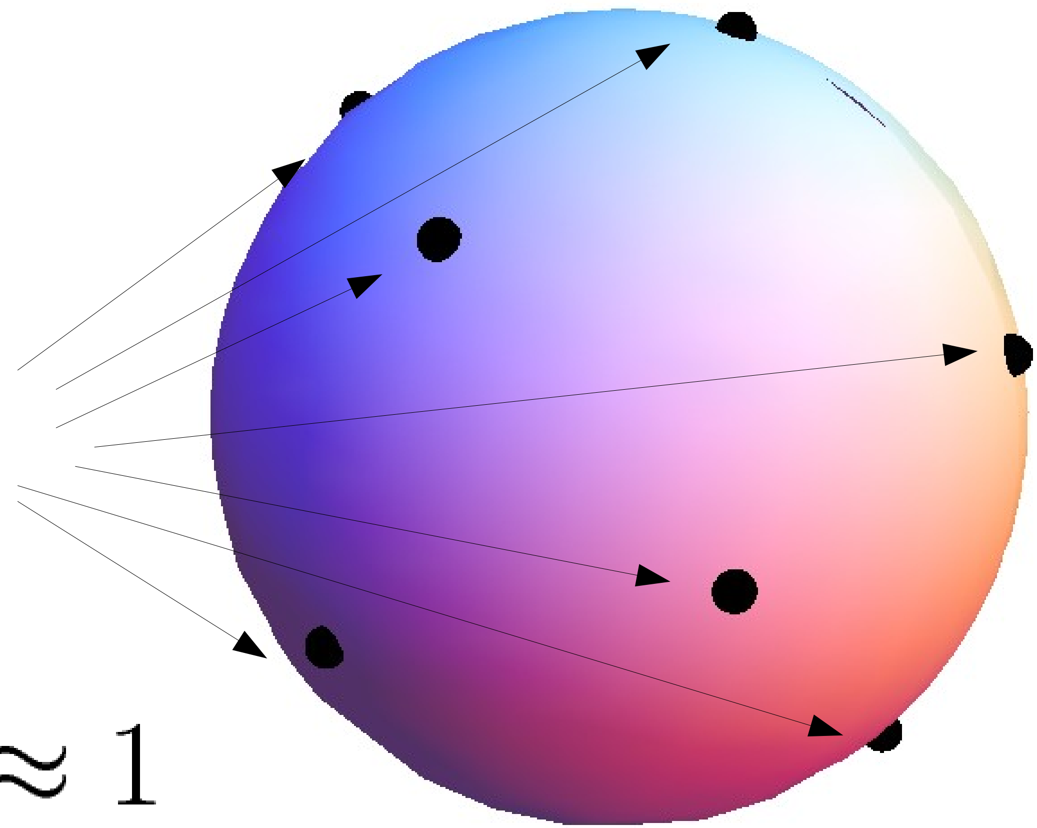
$$\sum_{i=1}^{12} f(R\hat{u}_i) = const.$$ if and only if

$$f(\hat{u}) = \hat{u} \cdot A\hat{u}$$

# Higher Dimensions

Q: When is a lattice packing of hyperspheres locally best?

Contact points: S



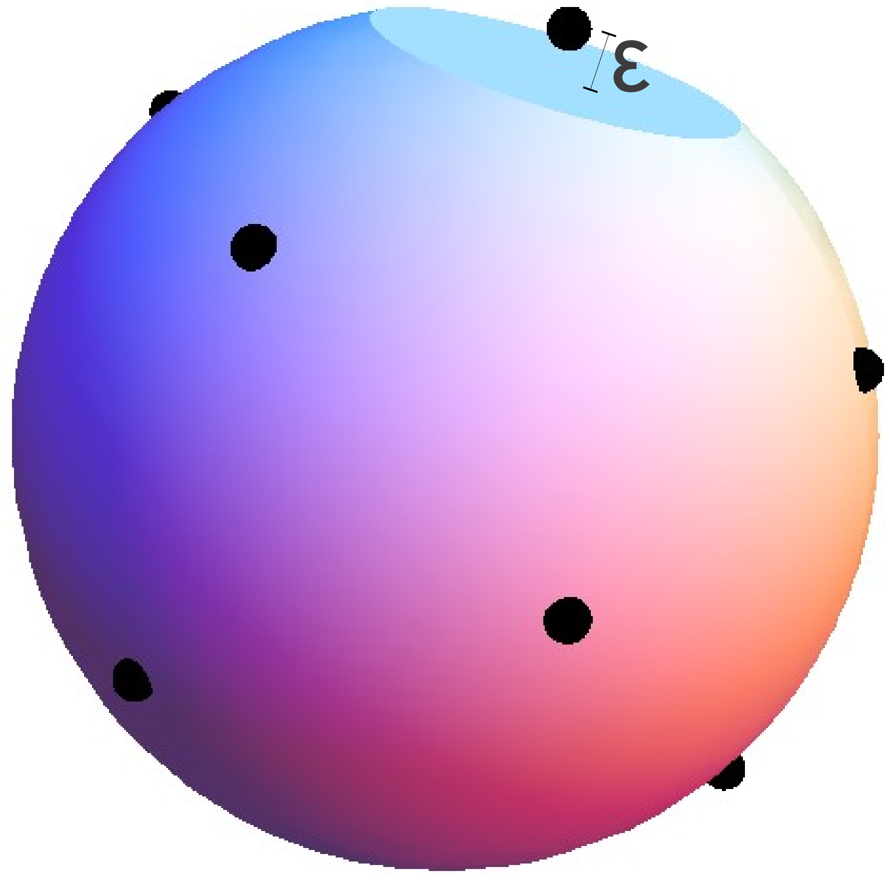
A: If and only if for $T \approx 1$

$$||T\mathbf{x}|| \geq ||\mathbf{x}|| \text{ for all } \mathbf{x} \in S \implies \det T > 1$$

S is “perfect and eutactic”

# Higher Dimensions

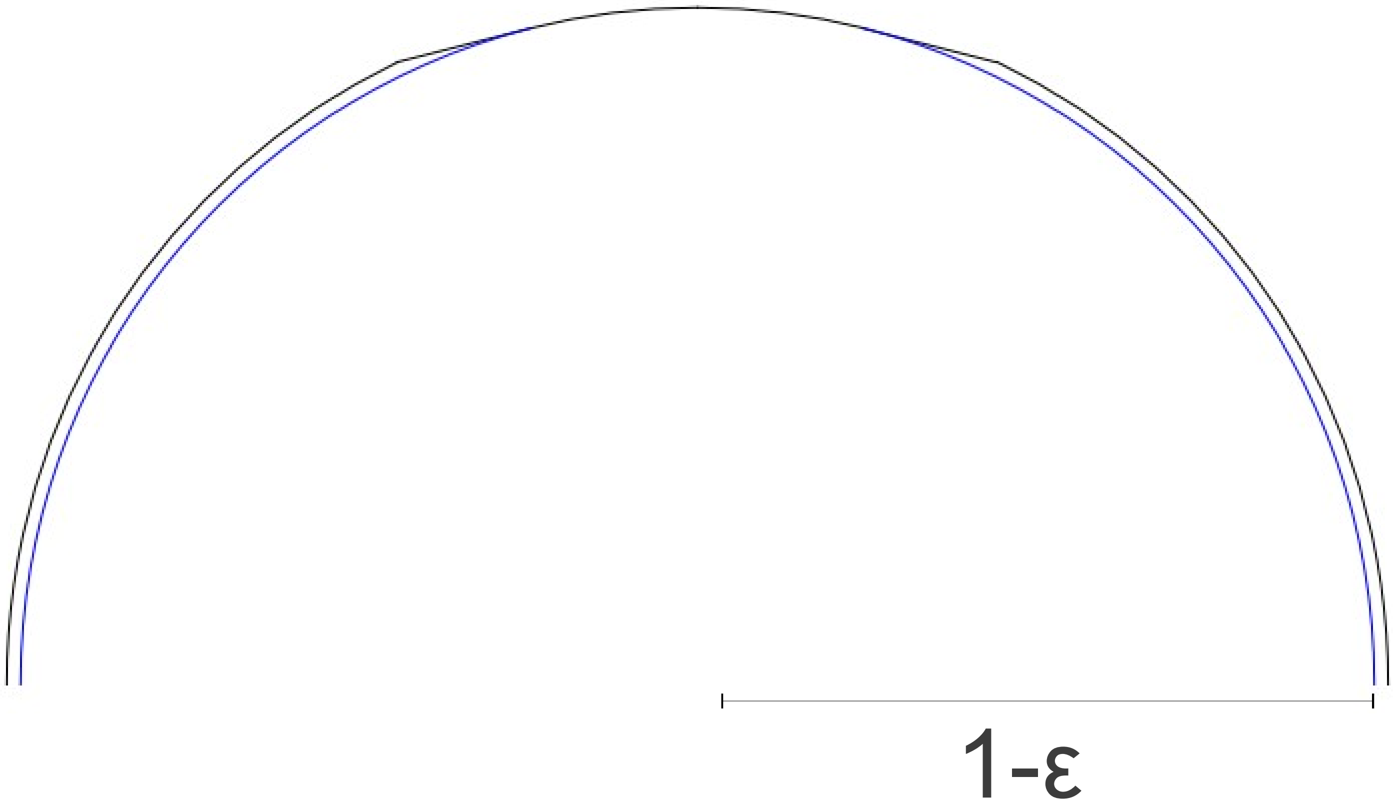
## Packing of a dented hypersphere




For d = 6, 7, 8, 24 the configuration of minimal vectors is redundantly perfect and eutactic. Therefore, the d-ball is reducible.

# Higher Dimensions

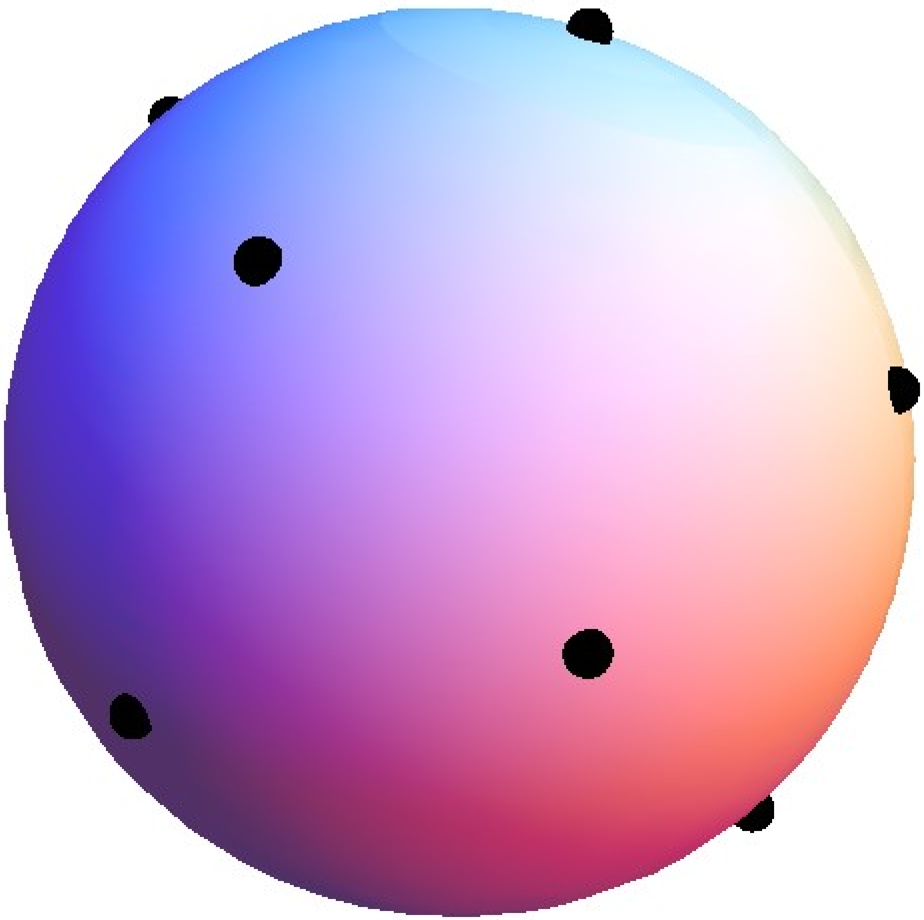
## Packing of a dented hypersphere




For d = 4, 5, the d-ball is irreducible, but the optimal number density of a dented d-ball rises only as the square of the depth of the dent.

# Higher Dimensions

## Packing of a shaved hypersphere

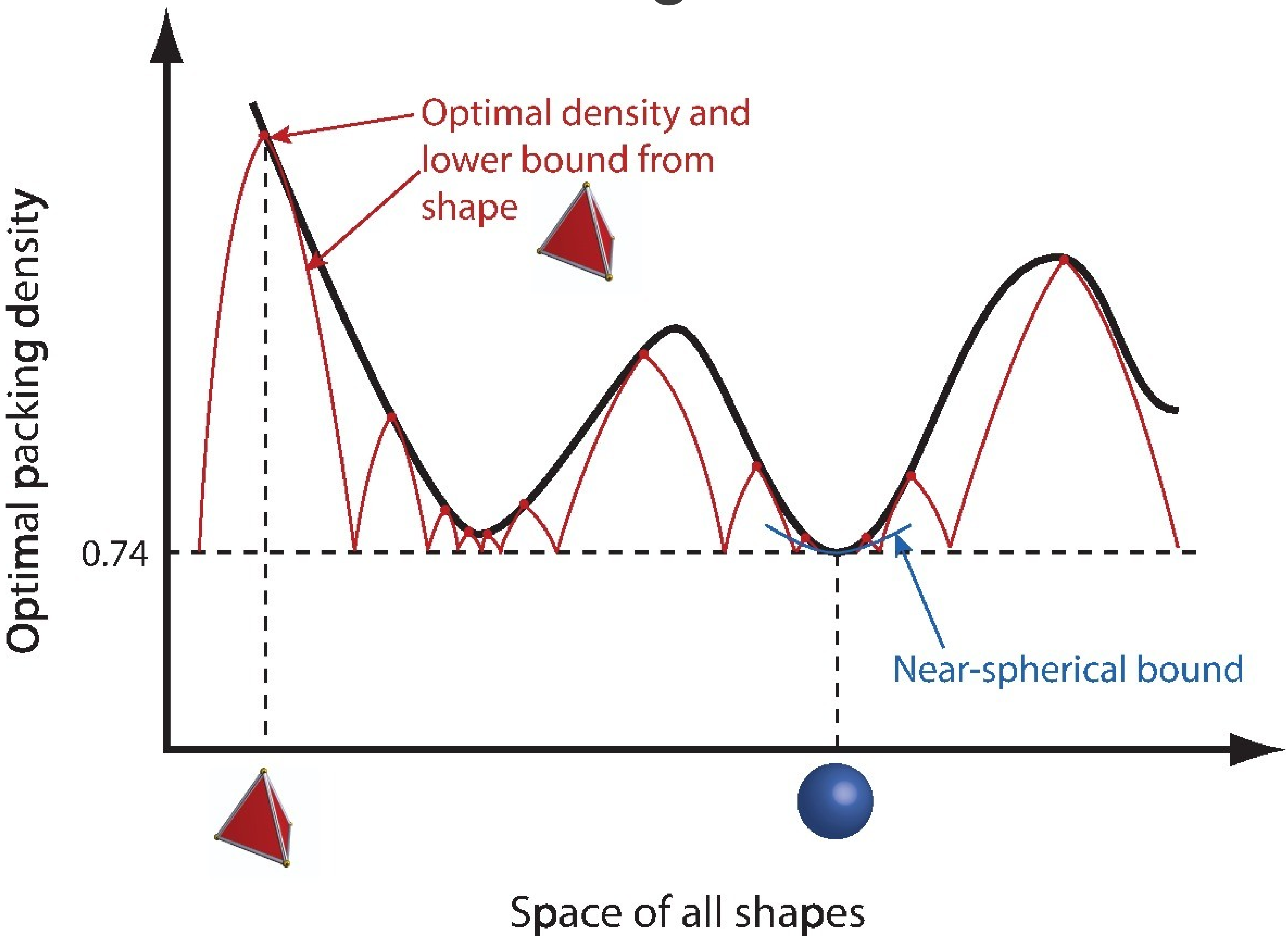


$$\frac{\rho(K)-\rho(B)}{\rho(B)} \sim \epsilon^2$$

$$\frac{V(B)-V(K)}{V(B)} \sim \epsilon$$

Therefore, for d=4, 5 the d-ball is irreducible but is not a local minimum of $\phi_L(K)$

# From local result to global result



# From local result to global result