Bravais New World

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Rogers 1958: "Many mathematicians believe, and all physicists know, that the density cannot exceed $\pi/\sqrt{18} = 0.7404\ldots$"

This talk: more things physicists "know" — crystallization transition, random close packing

Terminology:

Math.	Phys.	
"Lattice"	"Bravais lattice"	$L = A\mathbb{Z}^n$
"periodic pt. config."	"Lattice (w/a basis)"	$L = \bigcup_i (A\mathbb{Z}^n + \vec{t}_i)$
"(Delone) point config."	"point config."	

Auguste Bravais in 1848 classified the 14 Bravais lattices $\subseteq$ 230 space groups in 3D

For now, forget about lattices
What physicists know about point configurations.

Rogers talking about point configurations, since lattice problem in 3D was already solved

Hard sphere fluid

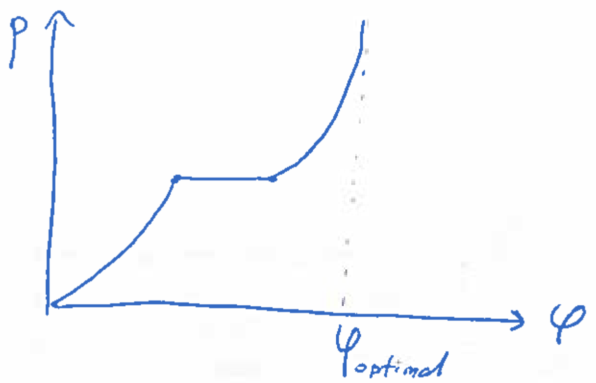
for $V \subseteq \mathbb{R}^n$, $\Omega_N(V) = \{(\vec{x}_i)_{i=1,\ldots N} \in V^N : \|\vec{x}_i - \vec{x}_j\| \geq 2\}$

$$S_N(V) = \log|\Omega_N(V)|$$

$$s(\varphi) = \lim_{\substack{N|B|/|V| \to \varphi \\ V \to \mathbb{R}^n}} \frac{1}{N} S_N(V)$$

$$P_N(V) = \frac{\partial S_N(V)}{\partial |V|} \longrightarrow p(\varphi) = \frac{1}{|B|} \frac{ds(\varphi)}{d(1/\varphi)}$$

$$P_N(V) \sim \left\langle \substack{\#\text{ of points} \\ \text{near } \partial V} \right\rangle$$

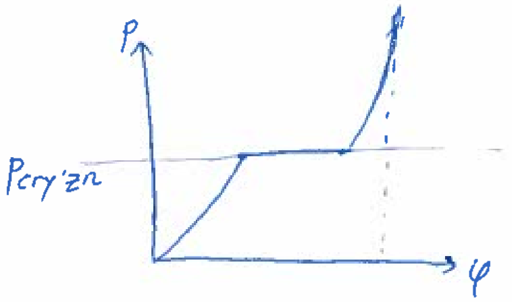


in 3D $\varphi_{opt} = 0.7404\ldots$

Above, we fixed N, V.
Alternatively, we can fix N, p

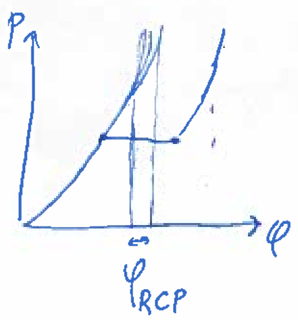
Configuration space: $\bigcup_{\alpha \in \mathbb{R}^+} \Omega_N(\alpha V)$

Measure: $\mu_{N,p}(\Omega') = \int_{\Omega'} d\alpha \, d^N\vec{x} \, e^{-p|\alpha V|}$



Long range order:

Metastable fluid:



in 3D $\varphi_{RCP} \approx 0.63 - 0.64$

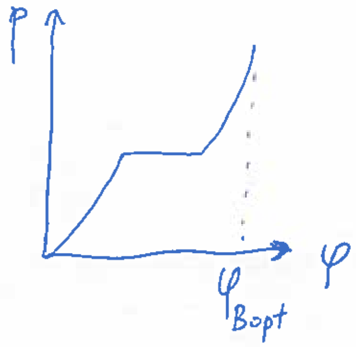
Numerical exploration is in moderate dimensions hard, requires very large number of spheres.

Solution: restrict to lattices

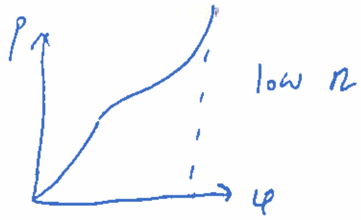
$G \in \Omega \subseteq S^n_{\geq 0}$ $\Omega \cong S^n_{\geq 0} / GL_n(\mathbb{Z})$

$\mu(\Omega') = \int_{\Omega'} d\mu(G) \, e^{-p(\det G)^{1/2}}$

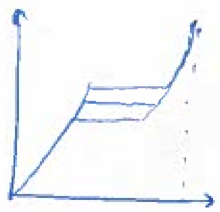
Bravais fluid $n = 7, \dots, 24$



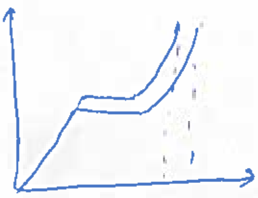
① crystallization transition
densest known lattice reproduced for $n \leq 20$



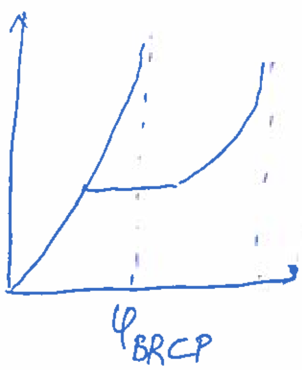
② No thermodynamic limit
→ no sharp transition
But transition sharpens for larger n



③ Hysteresis
different runs crystallize
at different pressures

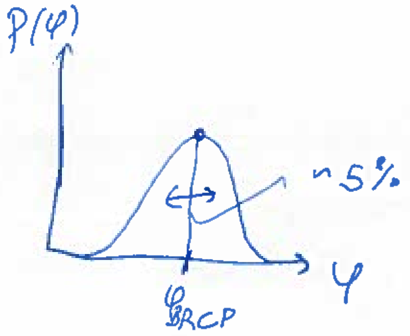


④ Some runs crystallize into
suboptimal (extreme) lattice
e.g. Λ_{11}' min $n = 4$ det: 1024
kissing #: 422
Λ_{11}^{min}: 432
Λ_{11}^{max}: 438
K_{11}: 432 det: 972

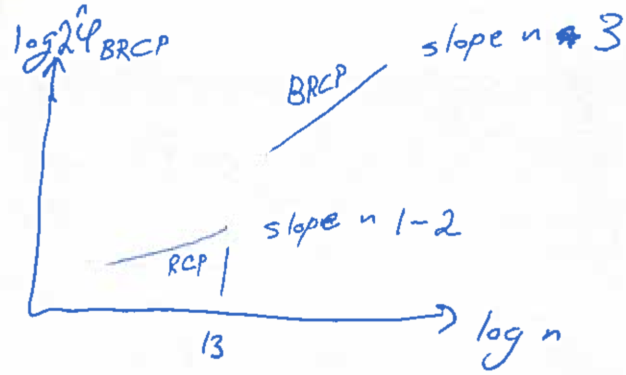


⑤ some runs (in low n, fast compression)
no crystallization at all

BRCP



~10,000 fast crunches for each $n = 15, \ldots 24$



we know $\varphi_{opt} \geq \varphi_{Bopt}$

but it seems $\varphi_{BRCP} \gg \varphi_{RCP}$

	RCP	BRCP
dim's accessible	$n \lesssim 13$	$14 \lesssim n \lesssim 24$
	$2^n \varphi \sim n^{1-2}$	$2^n \varphi \sim n^{\sim 3}$
isostaticity	$c = 2n$	$c = \frac{n(n+1)}{2}$
quasicontact singularity	✓	✓
theory	RSB: ~~[illegible]~~ $\varphi \sim n 2^{-n}$	? (Parisi '08)