Computational topology of configuration spaces

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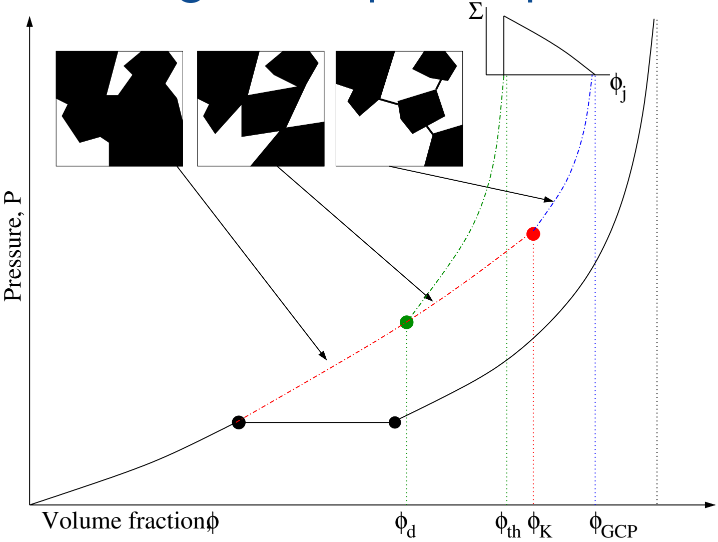
Stochastic Topology and Thermodynamic Limits
ICERM, Providence
October 17, 2016

Clustering of the phase space



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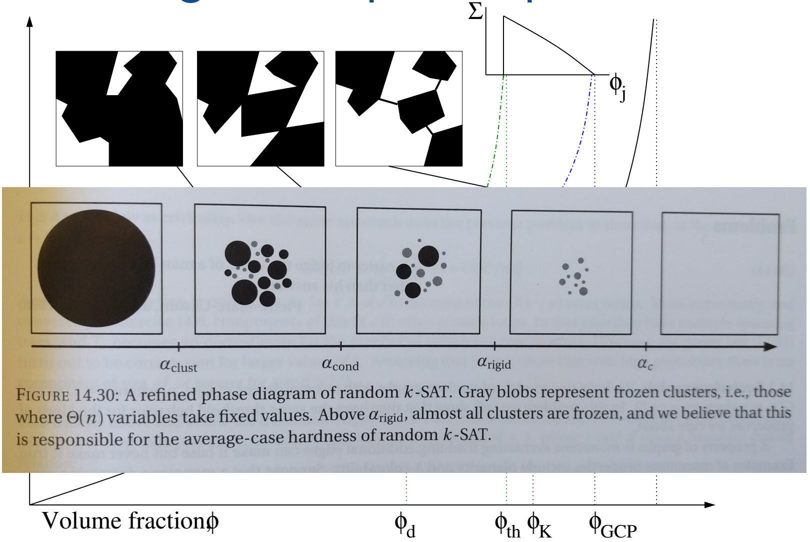

FIGURE 14.30: A refined phase diagram of random k-SAT. Gray blobs represent frozen clusters, i.e., those where $\Theta(n)$ variables take fixed values. Above α_{rigid}, almost all clusters are frozen, and we believe that this is responsible for the average-case hardness of random k-SAT.

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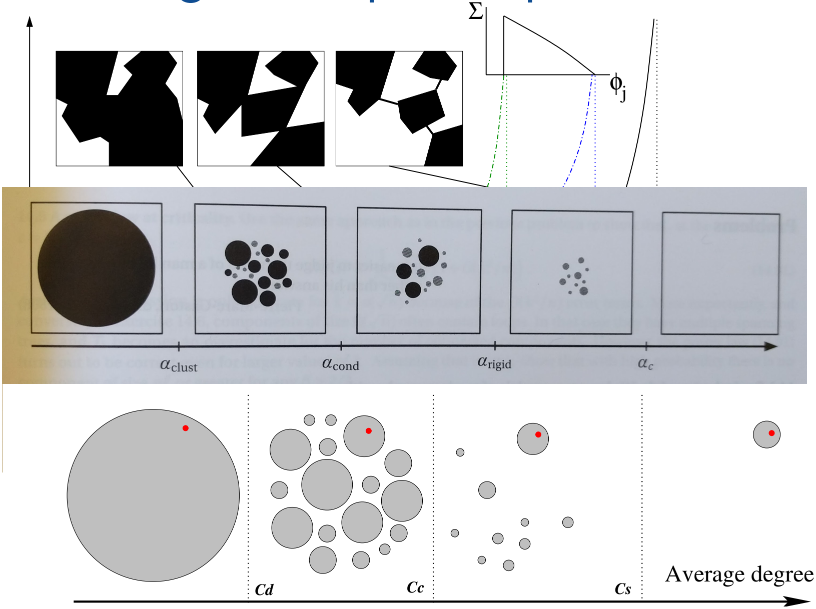


k-SAT clustering

Theorem

β, γ, θ, δ and $\epsilon_k \to 0$ exist such that for a random k-SAT formula with n variables and $m = \alpha n$ clauses, where

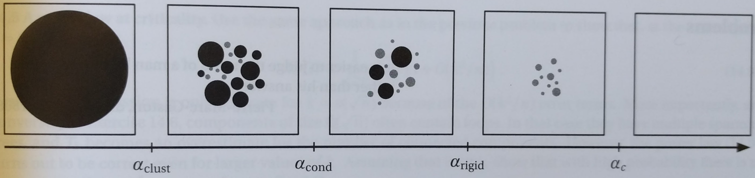
$$(1 + \epsilon_k)2^k \log(k)/k \leq \alpha \leq (1 - \epsilon_k)2^k \log(2)$$

the solution can is partitioned w.h.p. into clusters, s.t.

- *there are $\geq \exp(\beta n)$ clusters,*
- *any cluster has $\leq$ exp$(-\gamma n)$ of all solutions,*
- *solutions in distinct clusters are $\geq \delta n$ apart, and*
- *any connecting path violates $\geq \theta n$ clauses along it.*

Moore & Mertens, The Nature of Computation

Clustering phenomenology



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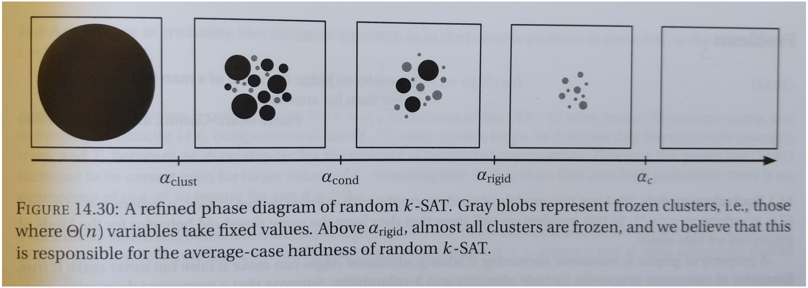

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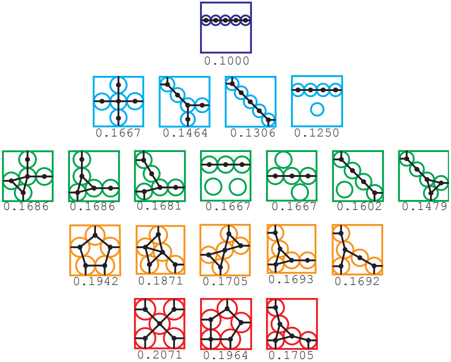
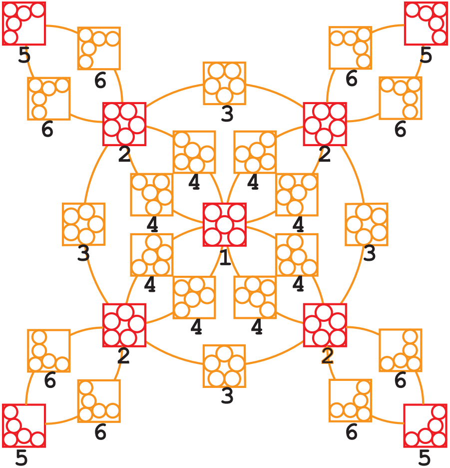
All about connected components of the configuration space. What about higher dimensional topological invariants?

Moore & Mertens, The Nature of Computation

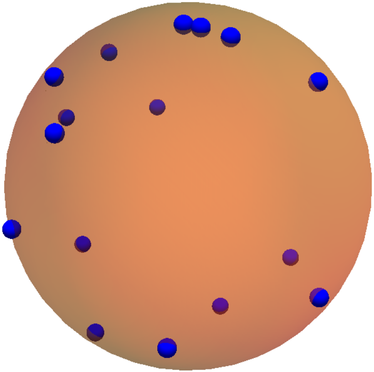
5 disks in a square





Carlsson, Gorham, Kahle, & Mason, Phys. Rev. E 85, 011303 (2012)

The perceptron



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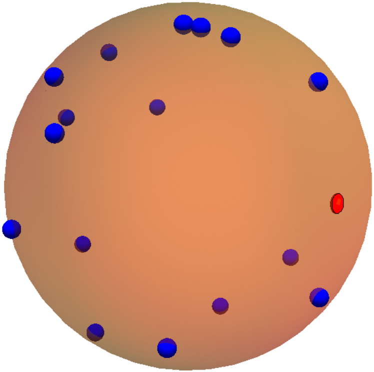

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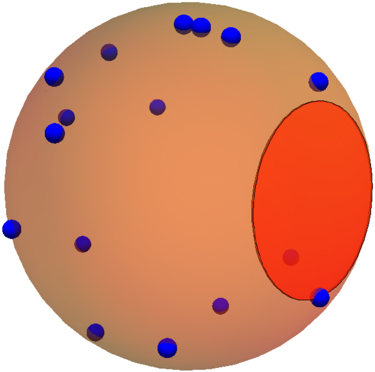

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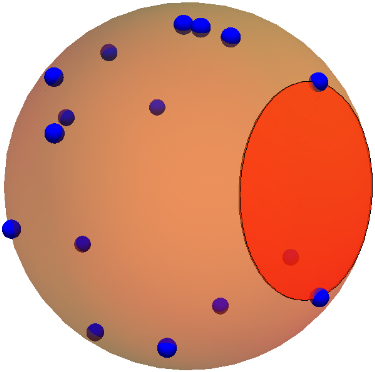

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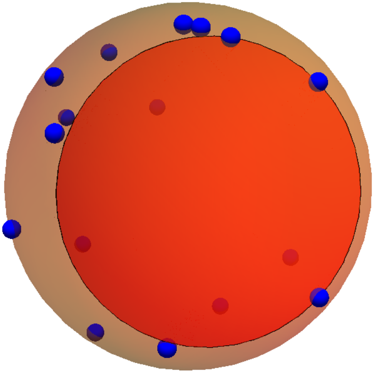

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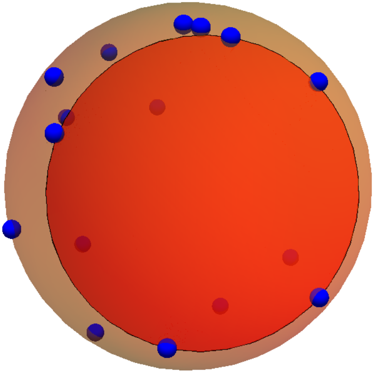

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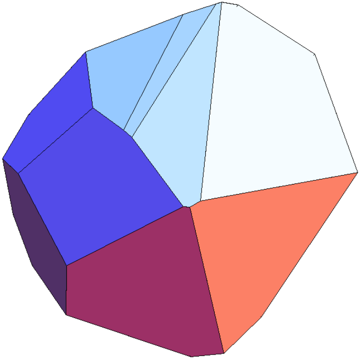

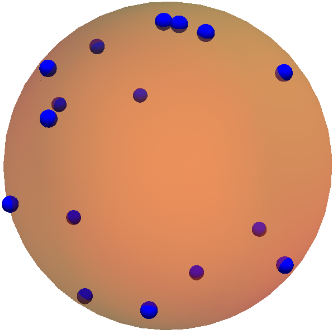

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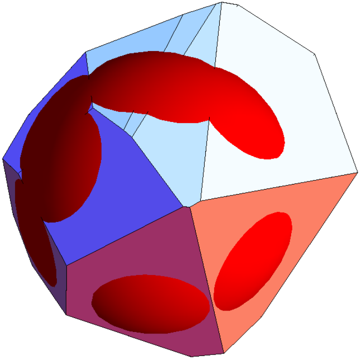

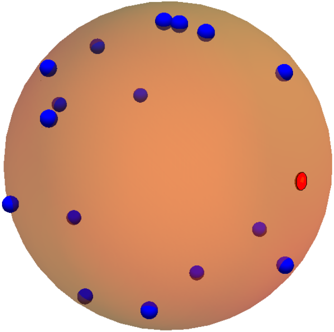

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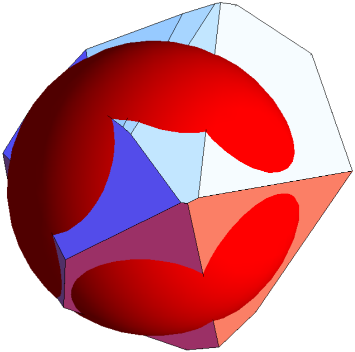

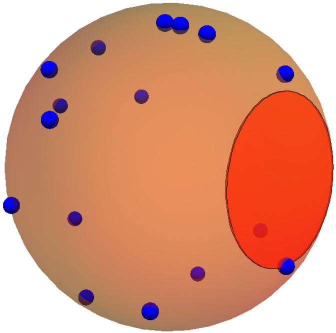

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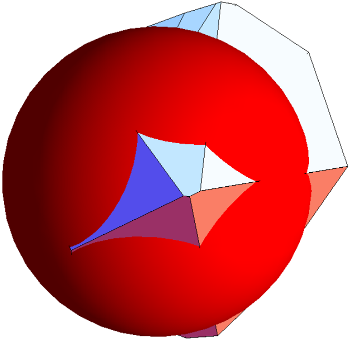
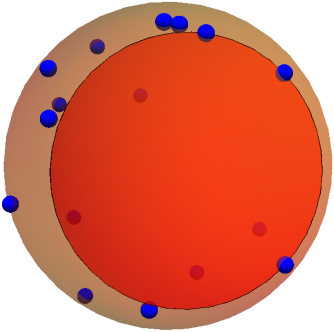
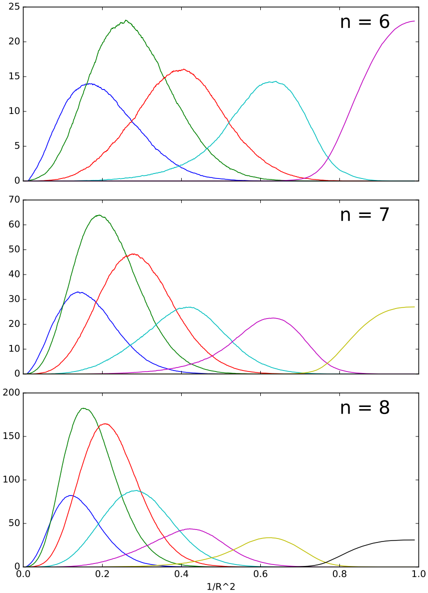
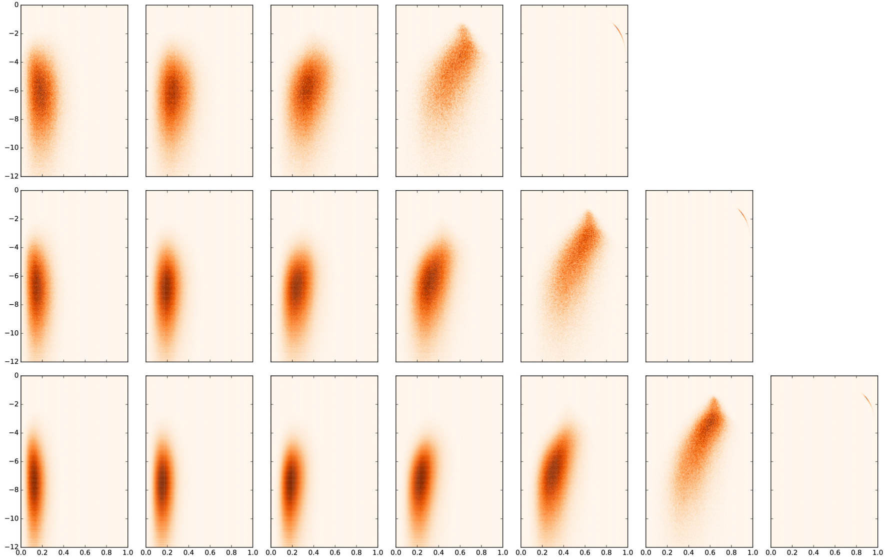




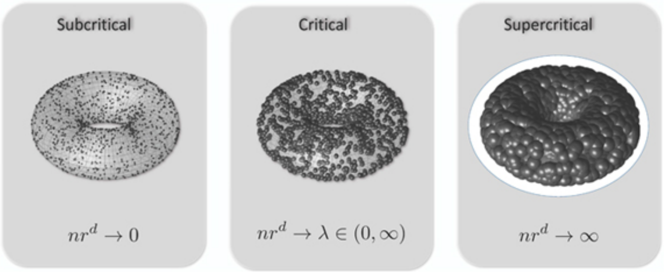
The perceptron – Betti numbers



The perceptron – persistence homology



Stochastic topology, a different criticality



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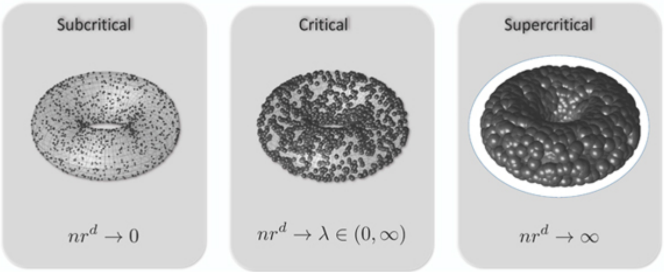
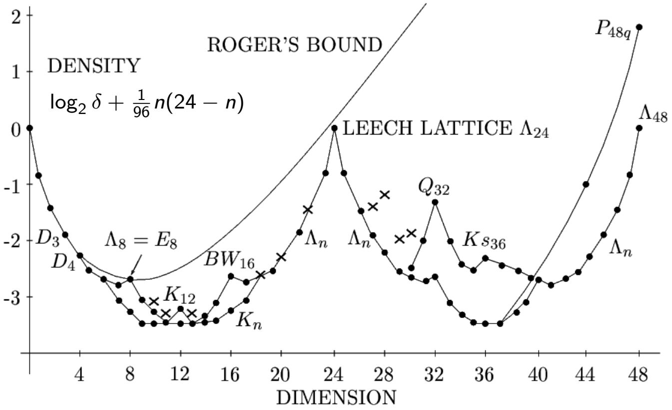


d fixed, $n \to \infty$: structure only at local scale

$d, n \to \infty$, $d \sim exp(n^2)$: no spatial structure

$d, n \to \infty$, $d \sim n$: structure at all scales

The sphere packing problem



Conway & Sloane, SPLAG

Packing problem restricted to lattices

Restricted to lattices, what is the densest packing structure?

n	L	
2	A_2	Lagrange (1773)
3	$D_3 = A_3$	Gauss (1840)
4	D_4	Korkin & Zolotarev (1877)
5	D_5	Korkin & Zolotarev (1877)
6	E_6	Blichfeldt (1935)
7	E_7	Blichfeldt (1935)
8	E_8	Blichfeldt (1935)
24	Λ_{24}	Cohn & Kumar (2004)

The space of lattices

$L = A\mathbb{Z}^n$

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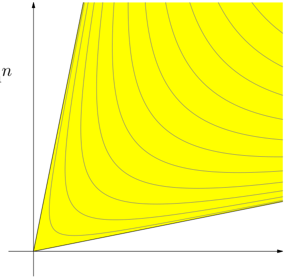
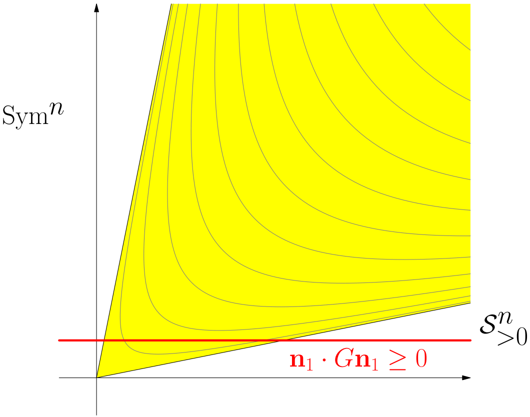
$O(n)\backslash GL_n(\mathbb{R}) = \mathcal{S}^n_{>0} \subset \mathrm{Sym}^n$, the space of symmetric, positive definite matrices: take $G = A^T A$.



The Ryshkov polyhedron

We are interested in the lattices with packing radius ≥ 1:
$\{G \in \mathcal{S}^n_{>0} : \mathbf{n} \cdot G\mathbf{n} \geq 1 \text{ for all } \mathbf{n} \in \mathbb{Z}^n\}$.



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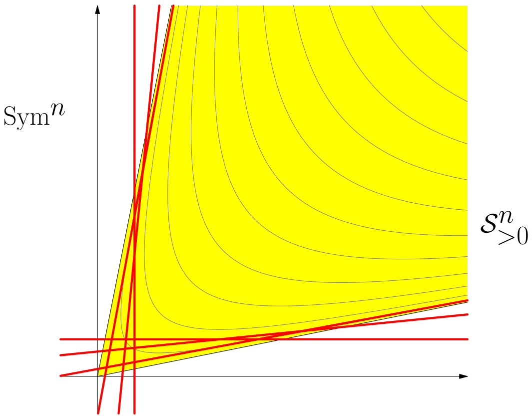

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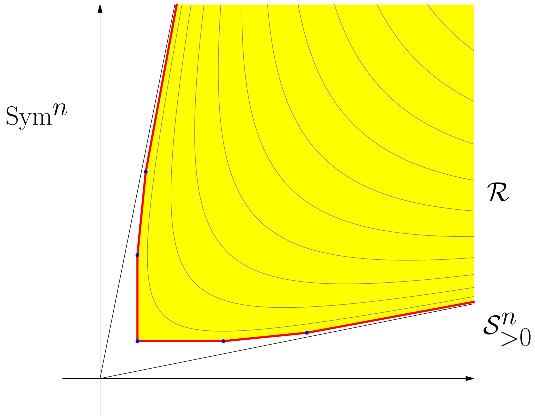
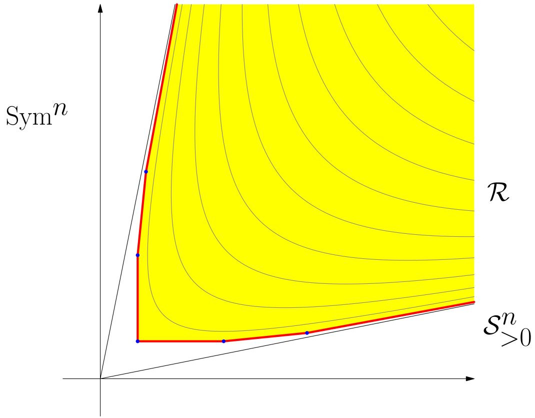


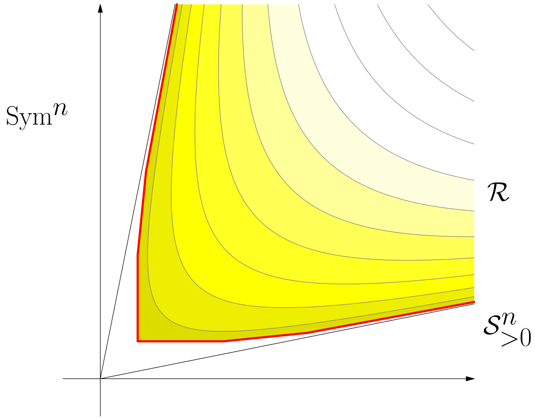
The Ryshkov polyhedron

The polytope is locally finite, and has finitely many faces modulo $GL_n(\mathbb{Z})$ action

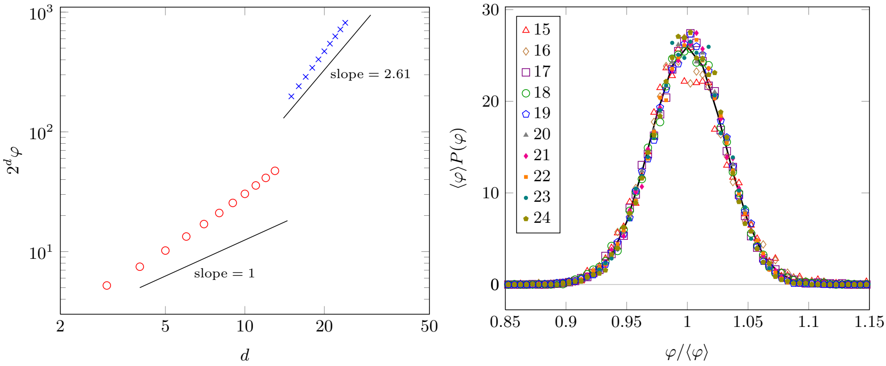


The Ryshkov polyhedron

We determinant (equivalently, density) gives a filtration of the space

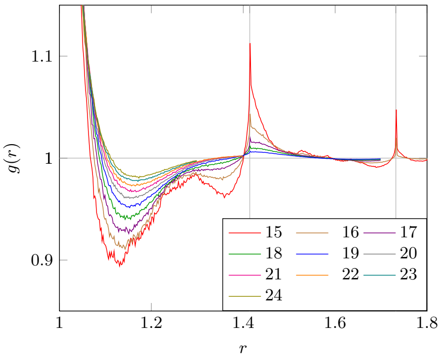


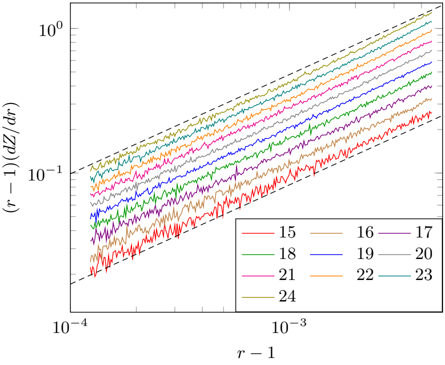
Lattice RCP



K, Marcotte, & Torquato, Phys. Rev. E 88, 062151 (2013)

Pair correlations and quasicontacts

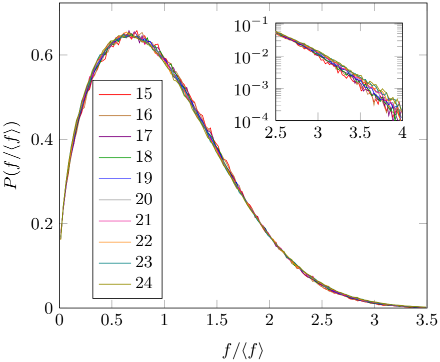


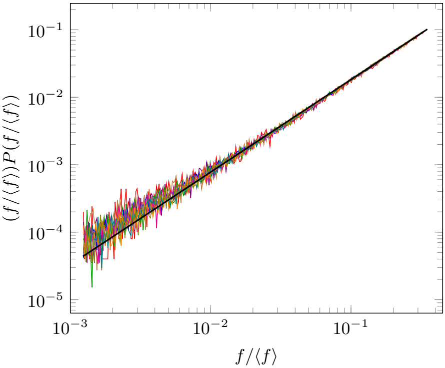


$$g(r) \sim (r-1)^{-\gamma}$$
$$Z(r) \sim d(d+1) + A_d(r-1)^{1-\gamma}$$
$$\gamma = 0.314 \pm 0.004$$

K, Marcotte, & Torquato, Phys. Rev. E 88, 062151 (2013)

Contact force distribution





$$P(f) \sim f^{\theta}$$
$$\theta = 0.371 \pm 0.010$$

K, Marcotte, & Torquato, Phys. Rev. E 88, 062151 (2013)

Topology of the space of lattices

PERFECT FORMS AND THE COHOMOLOGY OF MODULAR GROUPS 7

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\Sigma_n^\star(GL_5(\mathbb{Z}))$	2	5	10	16	23	25	23	16	9	4	3						
$\Sigma_n(GL_5(\mathbb{Z}))$					1	7	6	1	0	2	3						
$\Sigma_n^\star(GL_6(\mathbb{Z}))$		3	10	28	71	162	329	589	874	1066	1039	775	425	181	57	18	7
$\Sigma_n(GL_6(\mathbb{Z}))$						3	46	163	340	544	636	469	200	49	5		
$\Sigma_n^\star(SL_6(\mathbb{Z}))$		3	10	28	71	163	347	691	1152	1532	1551	1134	585	222	62	18	7
$\Sigma_n(SL_6(\mathbb{Z}))$			3	10	18	43	169	460	815	1132	1270	970	434	114	27	14	7

FIGURE 1. Cardinality of Σ_n and $\Sigma_n^\star$ for $N = 5, 6$ (empty slots denote zero).

n	6	7	8	9	10	11	12	13	14	15	16
$\Sigma_n^\star$	6	28	115	467	1882	7375	26885	87400	244029	569568	1089356
Σ_n				1	60	1019	8899	47271	171375	460261	955128

n	17	18	19	20	21	22	23	24	25	26	27
$\Sigma_n^\star$	1683368	2075982	2017914	1523376	876385	374826	115411	24623	3518	352	33
Σ_n	1548650	1955309	1911130	1437547	822922	349443	105054	21074	2798	305	33

FIGURE 2. Cardinality of Σ_n and $\Sigma_n^\star$ for $GL_7(\mathbb{Z})$.

Elbaz-Vincent, Gangl, & Soulé, Perfect forms and the cohomology of modular groups, arXiv:1001.0789

The perceptron – persistence homology