Numerical search methods for dense sphere packings in higher dimensions

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Asymptotic density at $1/p = 0$ for $d \to \infty$

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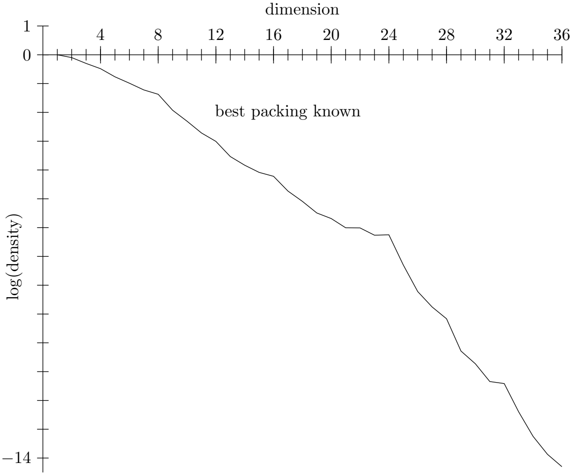
Minkowski: $\phi_{\text{opt}} \gtrsim 2^{-d}$
Others: $\phi_{\text{opt}} \gtrsim d2^{-d}$ for all d, and
$\phi_{\text{opt}} \gtrsim d \log(\log d) 2^{-d}$ for some subseq. $d \to \infty$.

Kabatiansky–Levenshtein: $\phi_{\text{opt}} \lesssim 2^{-0.5990d}$

Densest known packing for $d \leq 36$



Recently solved in $d = 8, 24$, still open for all other $d > 3$.

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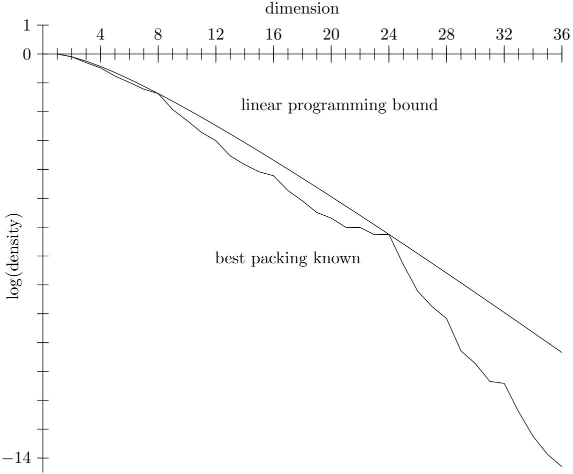
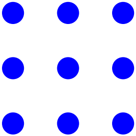
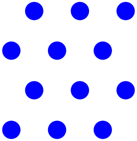
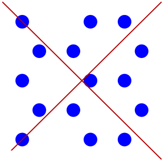
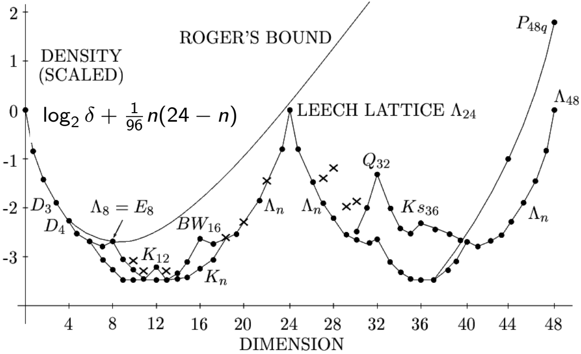


Bravais lattices



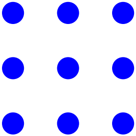




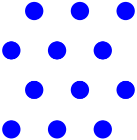


Bravais lattice is densest known packing in 26 out of first 36 dimensions.

Packings with fixed number of translational orbits



$m = 1$



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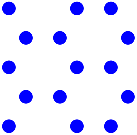
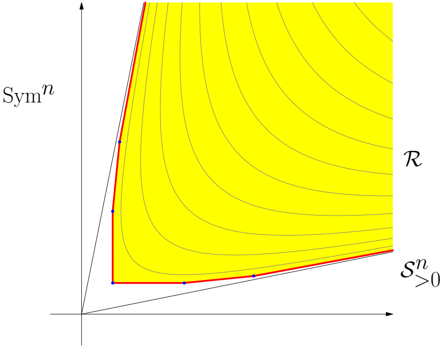


$m = 2$

$\lim_{m\to\infty} \phi_{d,m} = \phi_d$

Enumeration of locally optimal lattices

Lattice: $\Lambda = \{A\mathbf{n} : \mathbf{n} \in \mathbb{Z}^d\}$,
$A \sim d \times d$.
Let $G = A^T A$, then Λ has packing radius > 1 if $\mathbf{n}^T G \mathbf{n} \geq 2$ for all $\mathbf{n} \in \mathbb{Z}^d$.
$\det G$ is quasiconcave.



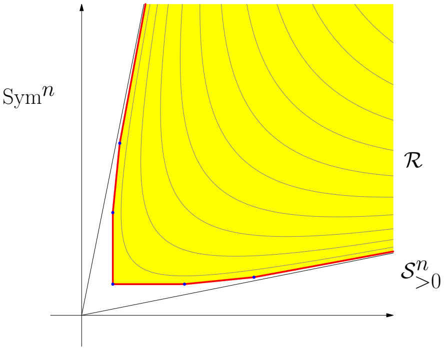
Also, invariant under $G \mapsto UGU^T$ when $U \in GL_n(\mathbb{Z})$.

Results of enumeration

dimension	2	3	4	5	6	7	8	9
# verts.	1	1	2	3	7	33	10916	$> 10^9$
# locally opt.	1	1	2	3	6	30	2408	

Enumeration of double-lattices ($m = 2$)

Double lattice: $\Lambda = \left\{ A\mathbf{n} : \mathbf{n} \in \mathbb{Z}^d \times \{0,1\} \right\}$, $A \sim d \times (d+1)$.
Let $G = A^T A$, then Λ has packing radius > 1 if $\mathbf{n}^T G \mathbf{n} \geq 2$ for all $\mathbf{n} \in \mathbb{Z}^d$.
$\det G_{1\ldots d, 1\ldots d}$ is quasiconcave.
But constraint $\operatorname{rank} G = d$ is nonlinear.



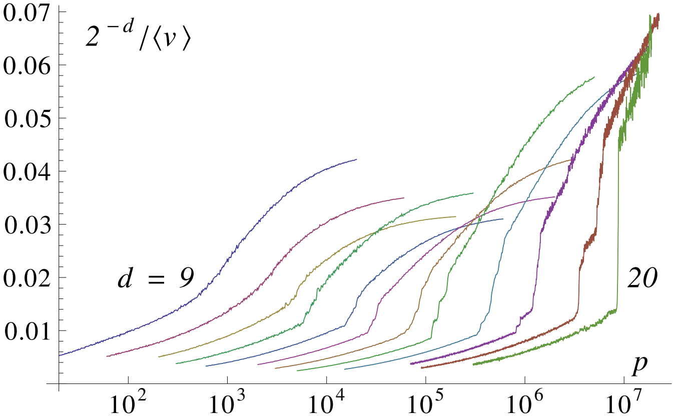
Nevertheless, can prove, all local optima live on edges.

Results of enumeration ($m = 2$)

dimension	3	4	5
# verts.	4	10	34
# rank-d pts. on edges	3 (1)	7 (3)	31+full edge (23)
# locally opt.	3 (1)	7 (3)	29 (20)
degen. of global opt.	3 (1)	2 (0)	5 (2)

Andreanov & K, in prep.

Thermodynamically sampling lattices

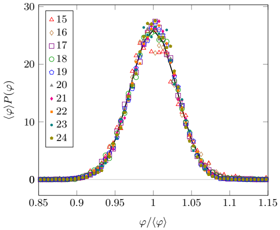


Prob. $\sim \exp(-pV_{\text{unit cell}})$

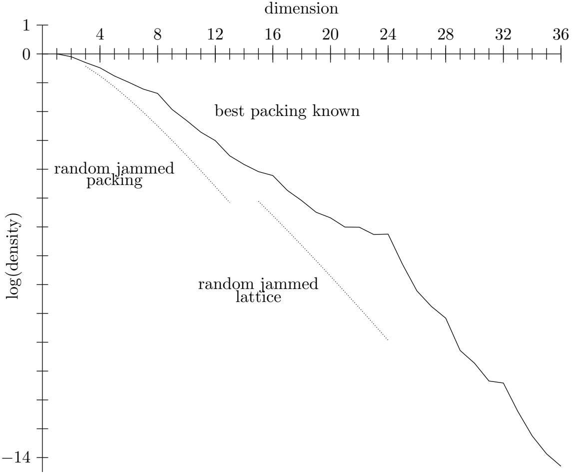
Densest known lattice recovered in some runs for $d \leq 20$

K, Phys. Rev. E 87, 063307 (2013)

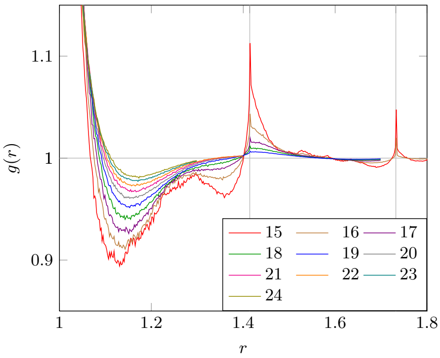
Lattice RCP

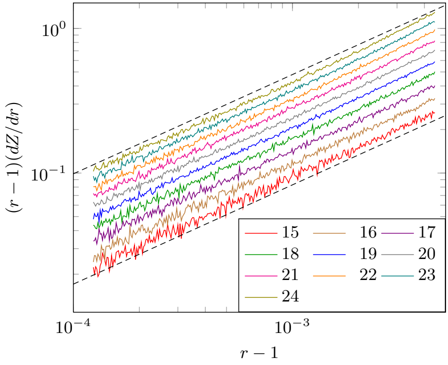


$\phi \sim d^{2.61} 2^{-d}$?



Pair correlations and quasicontacts

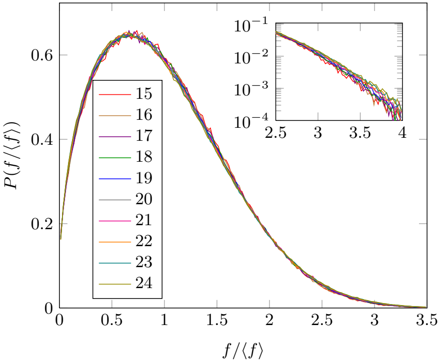


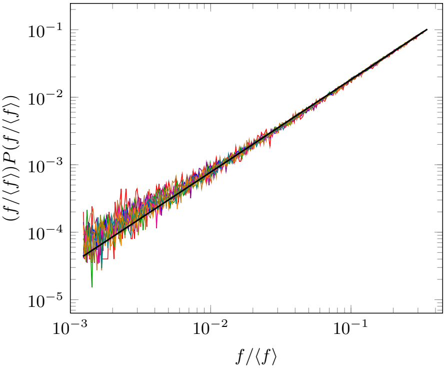


$$g(r) \sim (r-1)^{-\gamma}$$
$$Z(r) \sim d(d+1) + A_d(r-1)^{1-\gamma}$$
$$\gamma = 0.314 \pm 0.004$$

K, Marcotte, & Torquato, Phys. Rev. E 88, 062151 (2013)

Contact force distribution





$$P(f) \sim f^{\theta}$$
$$\theta = 0.371 \pm 0.010$$

K, Marcotte, & Torquato, Phys. Rev. E 88, 062151 (2013)

Future

- Cleverer annealing: got $d \leq 22$; $d = 23$ almost working.
- Full enumeration for higher d, m.
- Annealing for $m > 1$ (hope to discover packings denser than already known in some d).