Worst Packing Shapes

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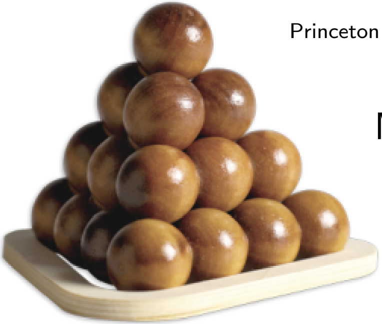
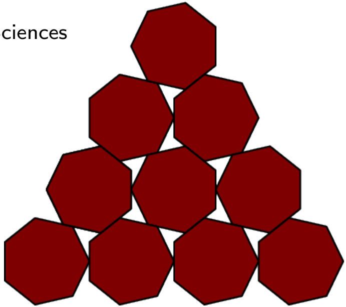


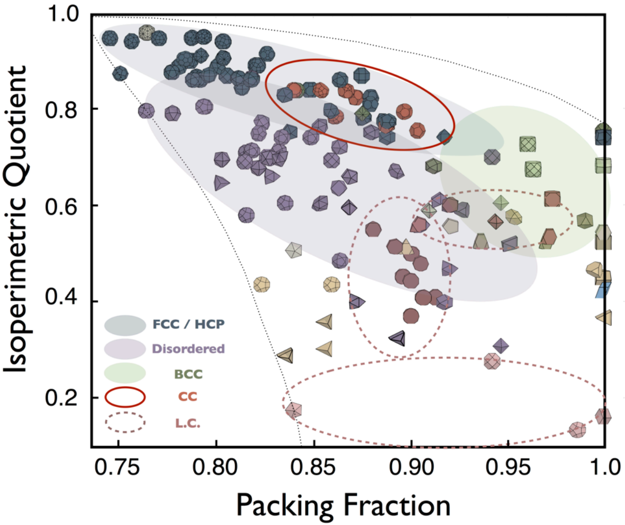
May 24, 2013



From Hilbert's 18$^{\text{th}}$ Problem

"How can one arrange most densely in space an infinite number of equal solids of a given form, e.g., spheres with given radii or regular tetrahedra with given edges, that is, how can one so fit them together that the ratio of the filled to the unfilled space may be as large as possible?"

Packing non-spherical shapes

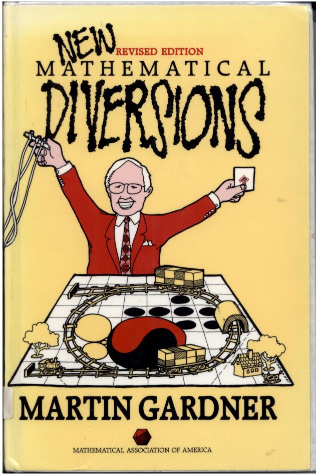


Damasceno, Engel, and Glotzer, 2012, unpublished.

The Miser's Problem

A miser is required by a contract to deliver a chest filled with gold bars, arranged as densely as possible. The bars must be identical, convex, and much smaller than the chest. What shape of bar should the miser cast so as to part with as little gold as possible?

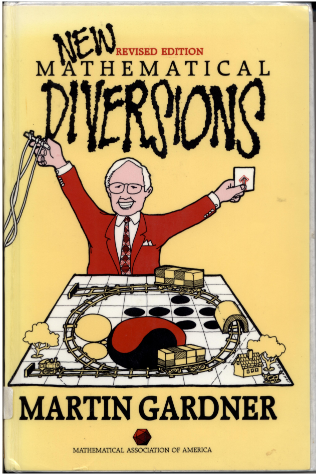
Ulam's Conjecture



"Stanislaw Ulam told me in 1972 that he suspected the sphere was the worst case of dense packing of identical convex solids, but that this would be difficult to prove."

1995 postscript to the column "Packing Spheres"

Ulam's Last Conjecture



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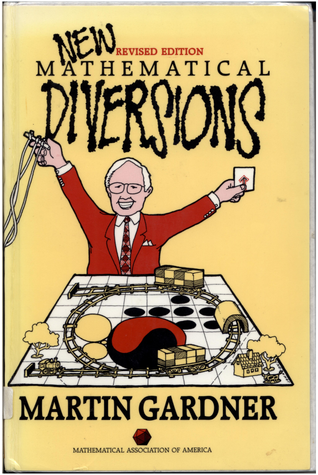

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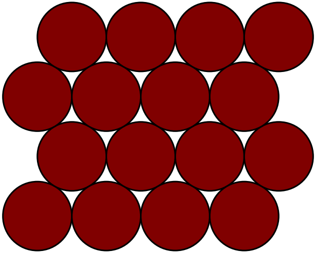
Naive motivation: sphere is the least free solid (three degrees of freedom vs. six for most solids).

1995 postscript to the column "Packing Spheres"

In 2D disks are not worst



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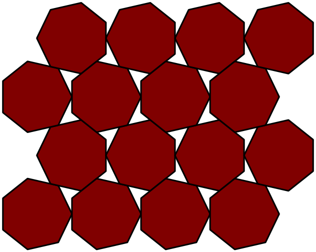

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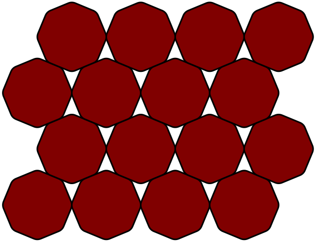
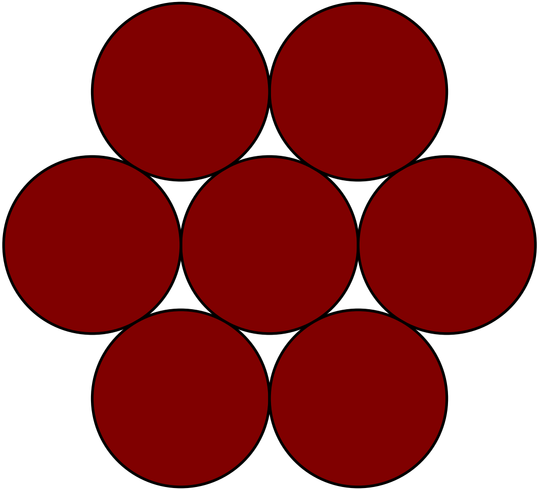


Why can we improve over circles?



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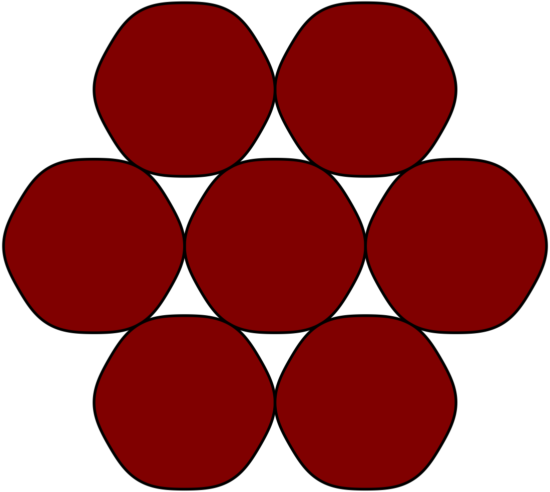

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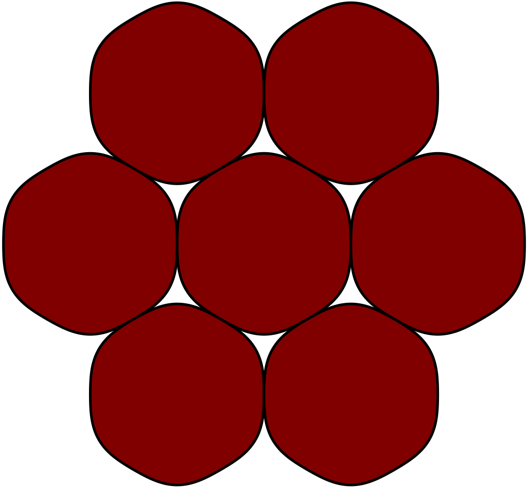

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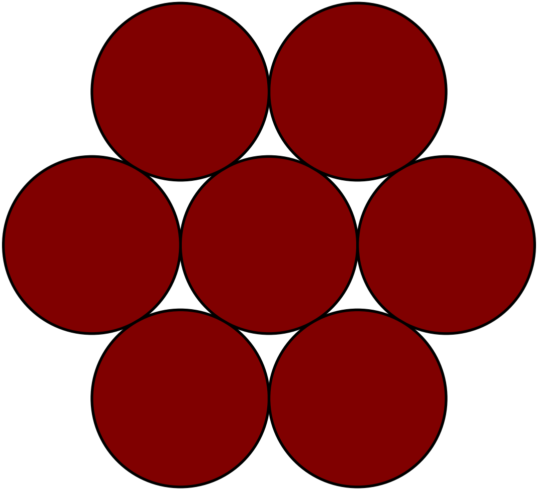

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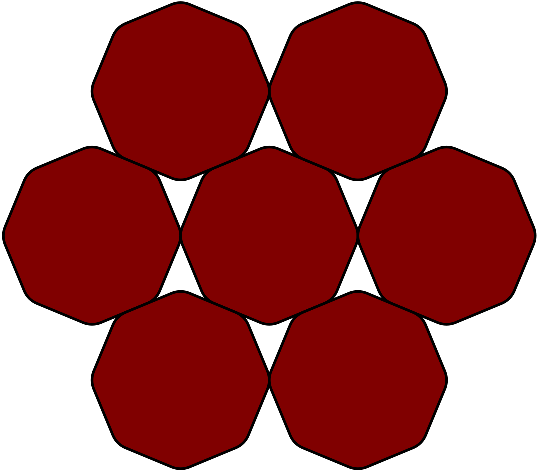

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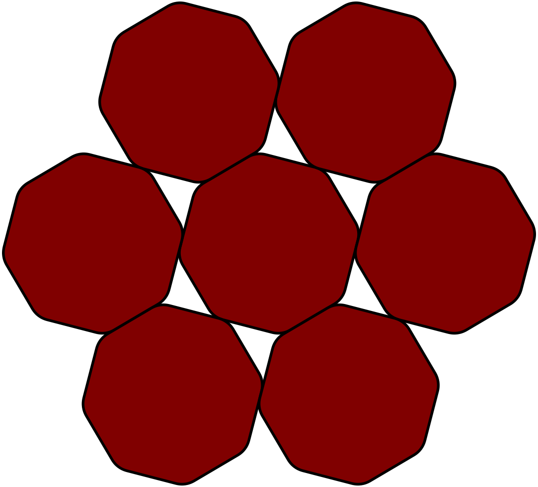

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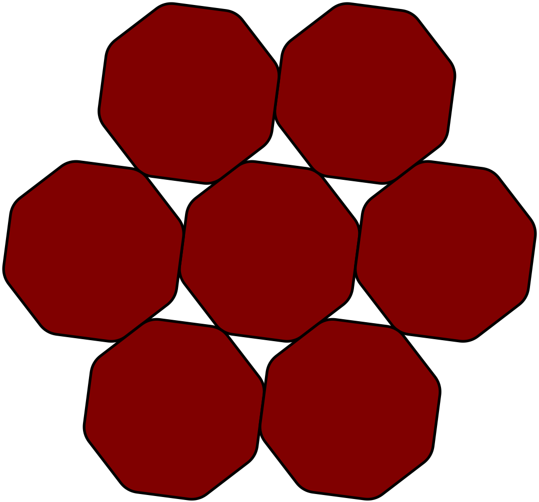

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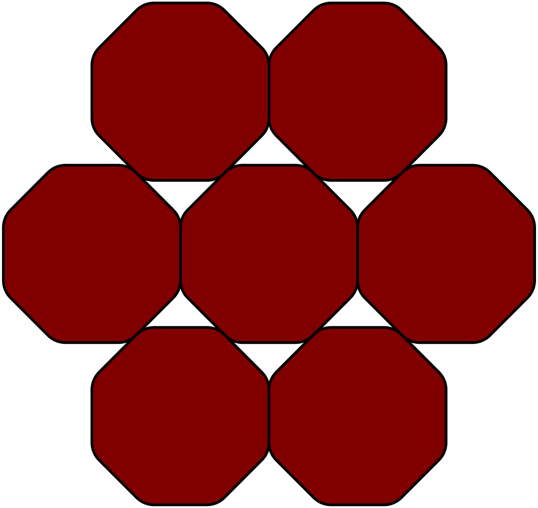

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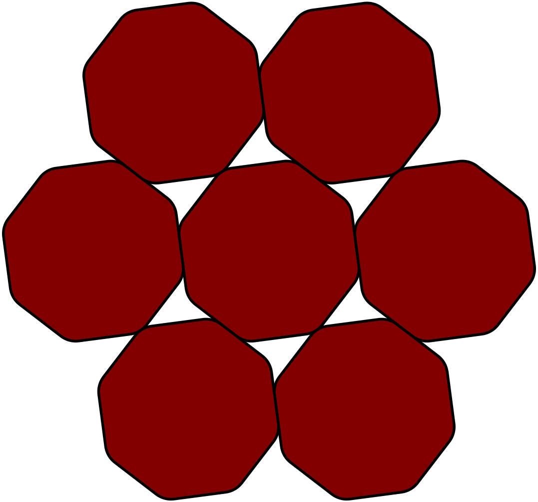

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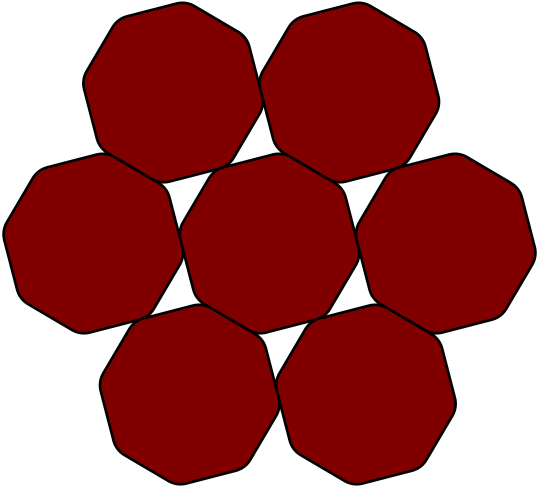

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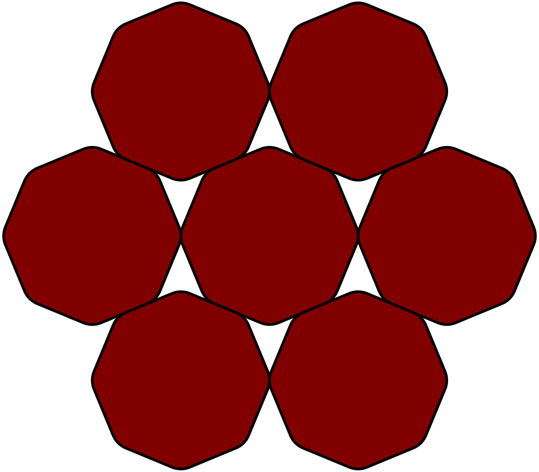

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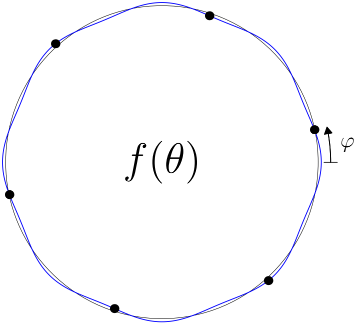

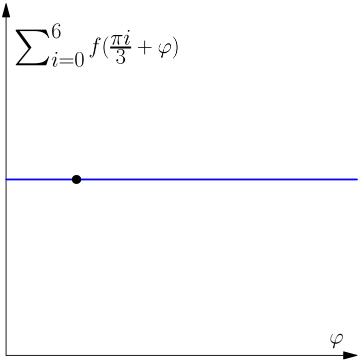

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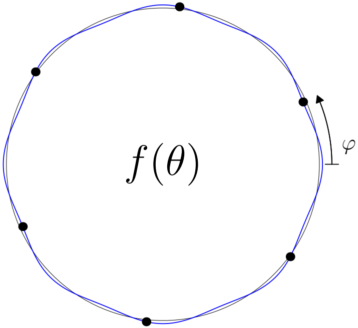

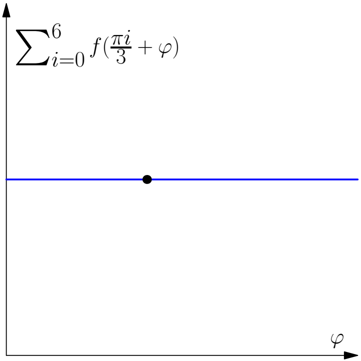

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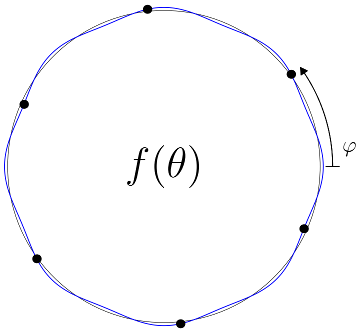

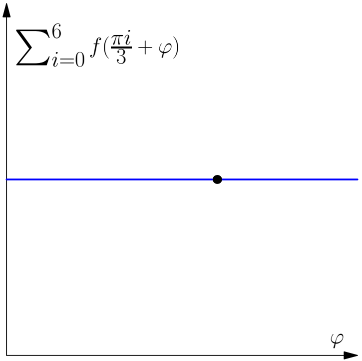

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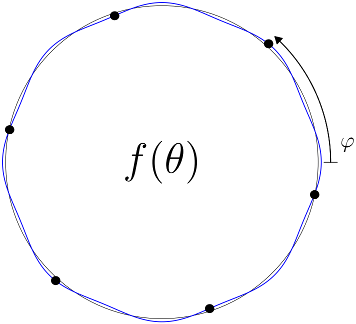

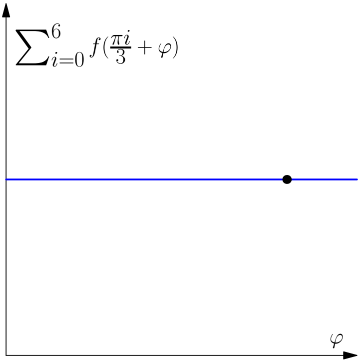

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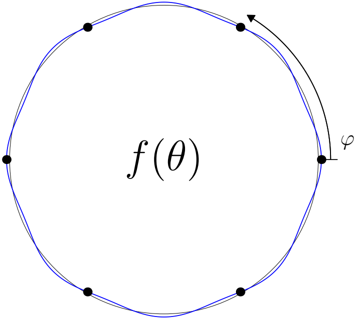

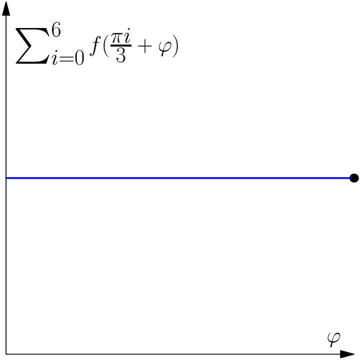

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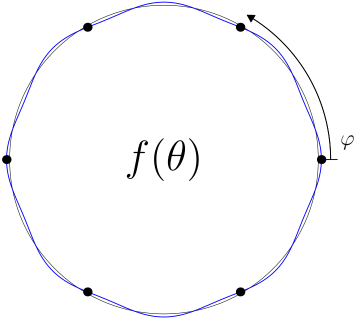
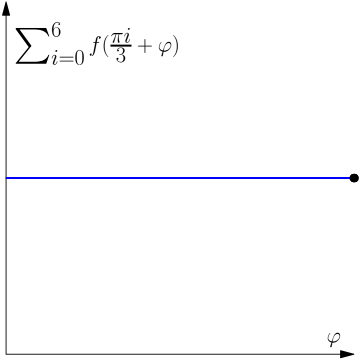
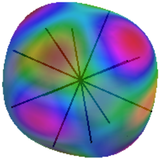
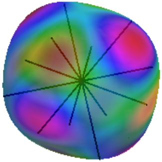


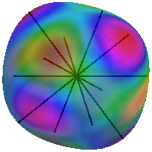


$$f(\theta) = 1 + \epsilon cos(8\theta)$$

Why can we not improve over spheres?







Lemma

Let f be an even function $S^2 \to \mathbb{R}$. $\sum_{i=1}^{12} f(R\mathbf{x}_i)$ is independent of R if and only if the expansion of $f(\mathbf{x})$ in spherical harmonics terminates at $l = 2$.

YK and F. Nazarov, arXiv:1212.2551

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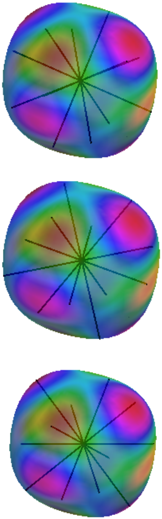

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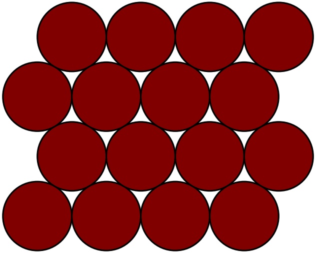
Theorem (YK, F. Nazarov)

The sphere is a local minimum of the optimal packing fraction among convex, centrally symmetric bodies.

YK and F. Nazarov, arXiv:1212.2551

In 2D disks are not worst



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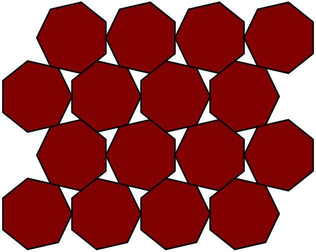

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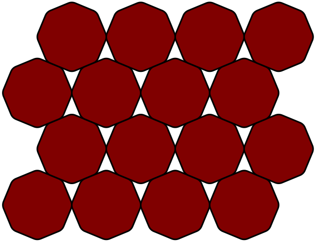
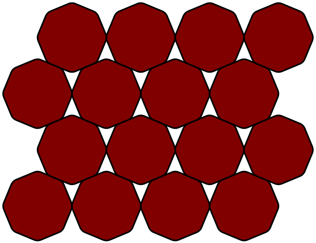


Reinhardt's conjecture



0.9024

Conjecture (K. Reinhardt, 1934)

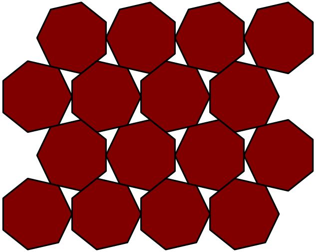
The smoothed octagon is an absolute minimum of the optimal packing fraction among convex, centrally symmetric bodies.

Theorem (F. Nazarov, 1986)

The smoothed octagon is a local minimum.

K. Reinhardt, Abh. Math. Sem., Hamburg, Hansischer Universität, Hamburg 10 (1934), 216
F. Nazarov, J. Soviet Math. 43 (1988), 2687

Regular heptagon is locally worst packing



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Theorem (YK)

Any convex body sufficiently close to the regular heptagon can be packed at a filling fraction at least that of the "double lattice" packing of regular heptagons.

Note: it is not proven, but highly likely, that the "double lattice" packing is the densest packing of regular heptagons.

YK, arXiv:1305.0289

Higher dimensions

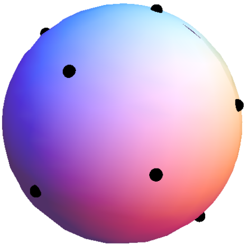
- In 2D, the circle is not a local minimum of packing fraction among c. s. convex bodies.
- In 3D, the sphere is a local minimum of packing fraction among c. s. convex bodies.
- What can we say about spheres in higher dimensions?

Higher dimensions

- In 2D, the circle is not a local minimum of packing fraction among c. s. convex bodies.
- In 3D, the sphere is a local minimum of packing fraction among c. s. convex bodies.
- What can we say about spheres in higher dimensions?

- Note that in $d > 3$ we do not know the densest packing of spheres.
- But we do know the densest *lattice* packing in $d = 4, 5, 6, 7, 8$, and 24.

Extreme Lattices



Contact points $S(\Lambda)$ of the optimal lattice.

A lattice Λ is *extreme* if and only if $||T\mathbf{x}|| \geq ||\mathbf{x}||$ for all $\mathbf{x} \in S(\Lambda) \implies \det T > 1$ for $T \approx 1$.

YK and F. Nazarov, arXiv:1212.2551

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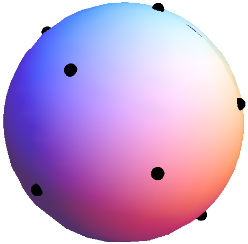

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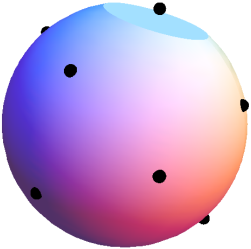
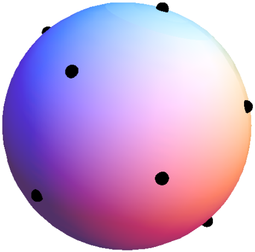


In $d = 6, 7, 8, 24$, the optimal lattice is *redundantly extreme*, and so the ball is *reducible*.

YK and F. Nazarov, arXiv:1212.2551

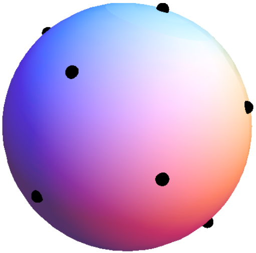
$d = 4$ and $d = 5$



In $d = 4, 5$, if $||T\mathbf{x}|| \geq ||\mathbf{x}||$ for all $\mathbf{x} \in S(\Lambda) \setminus \{\mathbf{x}_0\}$, and $||T\mathbf{x}_0|| > (1 - \epsilon)||\mathbf{x}_0||$, then $1 - \det T < C\epsilon^2$ (compared with $C\epsilon$ for $d = 2, 3$).

YK and F. Nazarov, arXiv:1212.2551

$d = 4$ and $d = 5$



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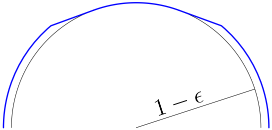

$$(\rho(K) - \rho(B))/\rho(B) \sim \epsilon^2$$
$$(V(B) - V(K))/V(B) \sim \epsilon$$

The ball is not a local minimum of the optimal packing fraction.

YK and F. Nazarov, arXiv:1212.2551

Summary of new results

- In $d = 2$, the heptagon is a local minimum of the optimal packing fraction, assuming the "double lattice" packing of heptagons is their densest packing.
- In $d = 3$, the ball is a local minimum of the optimal packing fraction among c. s. bodies.
- In $d = 4, 5$, the ball is not a local minimum of the optimal lattice packing fraction among c. s. bodies.
- In $d = 6, 7, 8, 24$, the ball is reducible with respect to lattice packing.
- For the lattice covering problem, the 3D ball is a local maximum of the optimal covering fraction, but the 4D and 5D balls are reducible.