The *Divide and Concur approach to packing*

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j/w:
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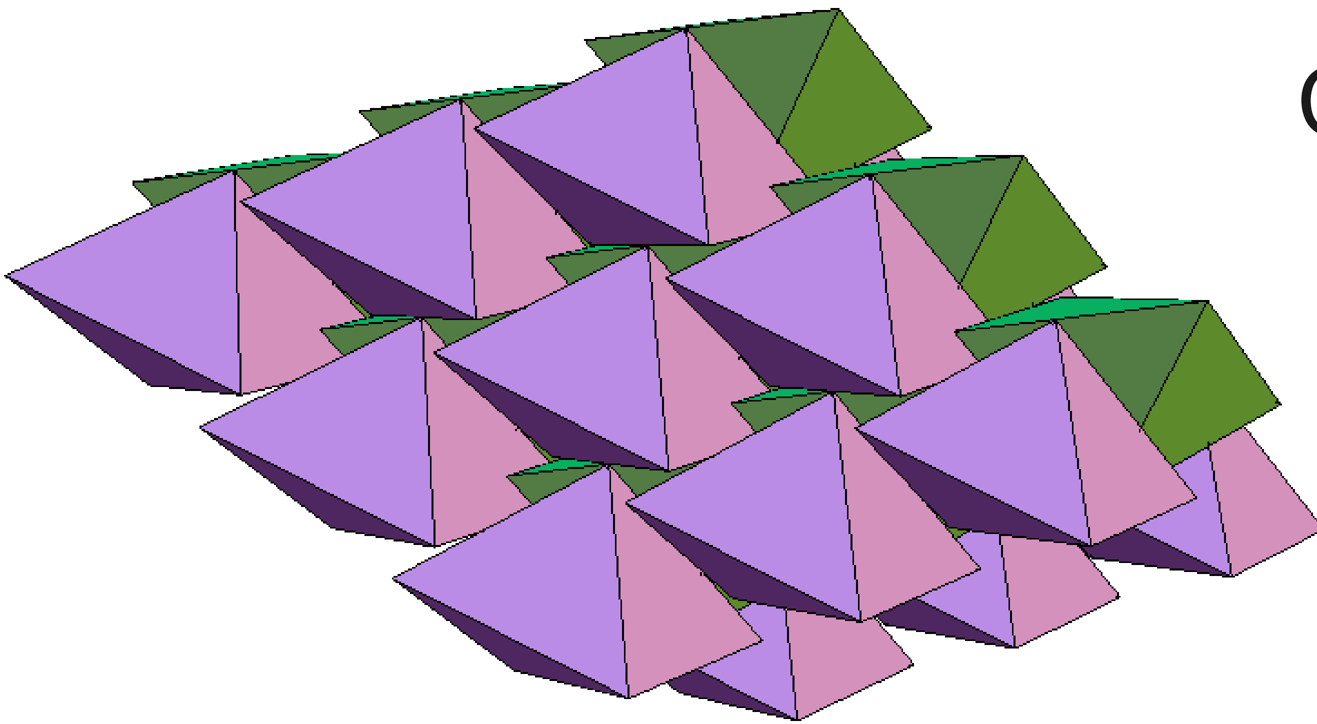
Simon Gravel



Shanks Conference
Vanderbilt University
May 17, 2010

Packing problems:

Optimization: given a collection of figures, arrange them without overlaps as densely as possible.

Feasibility: find an arrangement of density > φ

Possible computational approaches:

- Complete algorithm
- Specialized incomplete (heuristic) algorithm
- General purpose incomplete algorithm

  e.g.: simulated annealing, genetic algorithms, etc.

Divide and Concur belongs to the last category

Two constraint feasibility

$$x \in A \cap B$$

Example:

A = permutations of "`ailmopt`"

B = 7-letter English words

Two constraint feasibility

$$x \in A \cap B$$

Example:

A = permutations of “`ailmopt`”

B = 7-letter English words

x = “`optimal`”

More structure

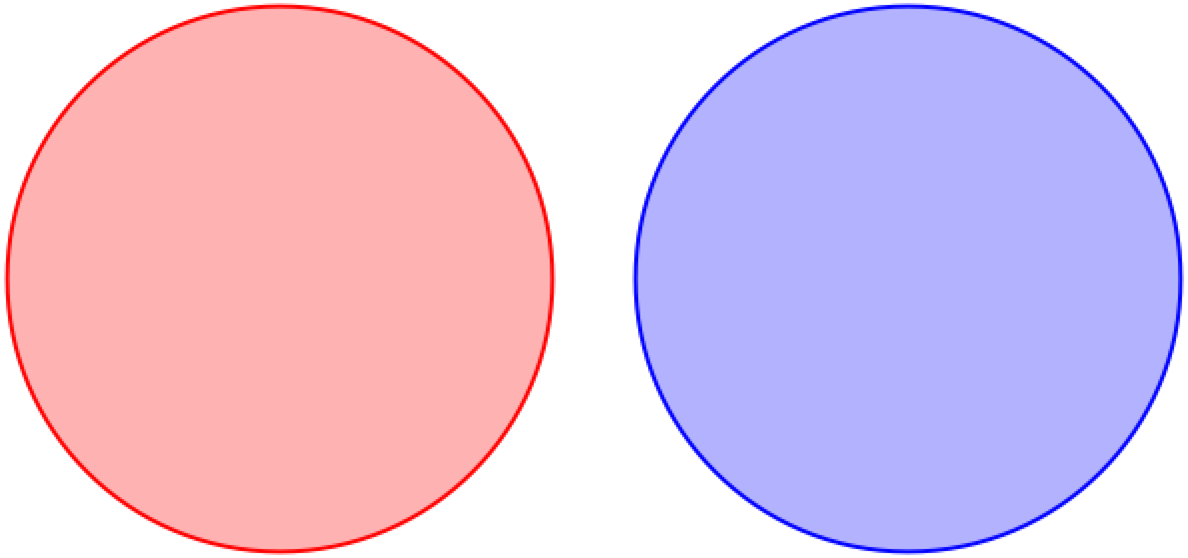
A, B are sets in a Euclidean configuration space Ω

simple constraints:

easy, efficient **projections** to A, B

$$P_A(x) = y \in A \quad \text{s.t.} \quad ||x - y|| \text{ is minimized}$$

Projection to the packing (no overlaps) constraint



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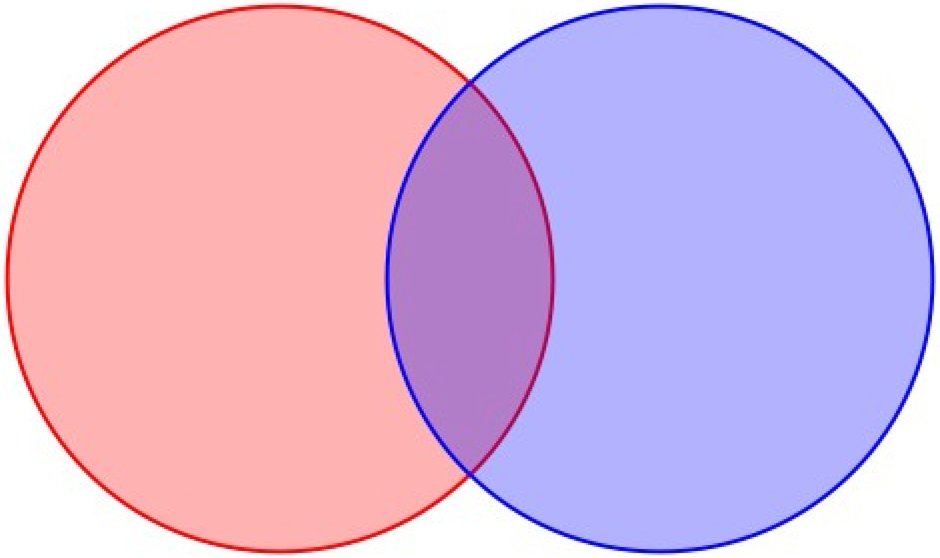

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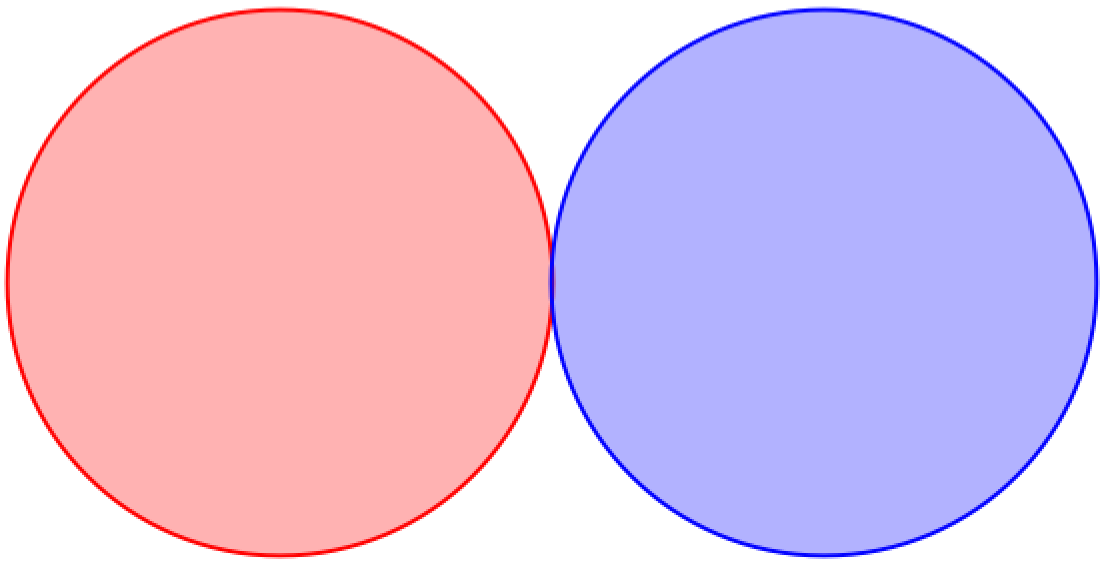
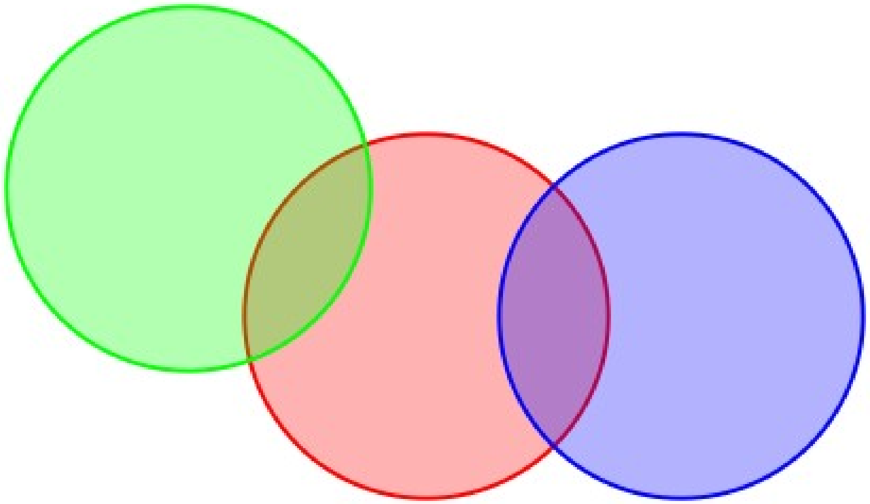


Dividing the Constraints



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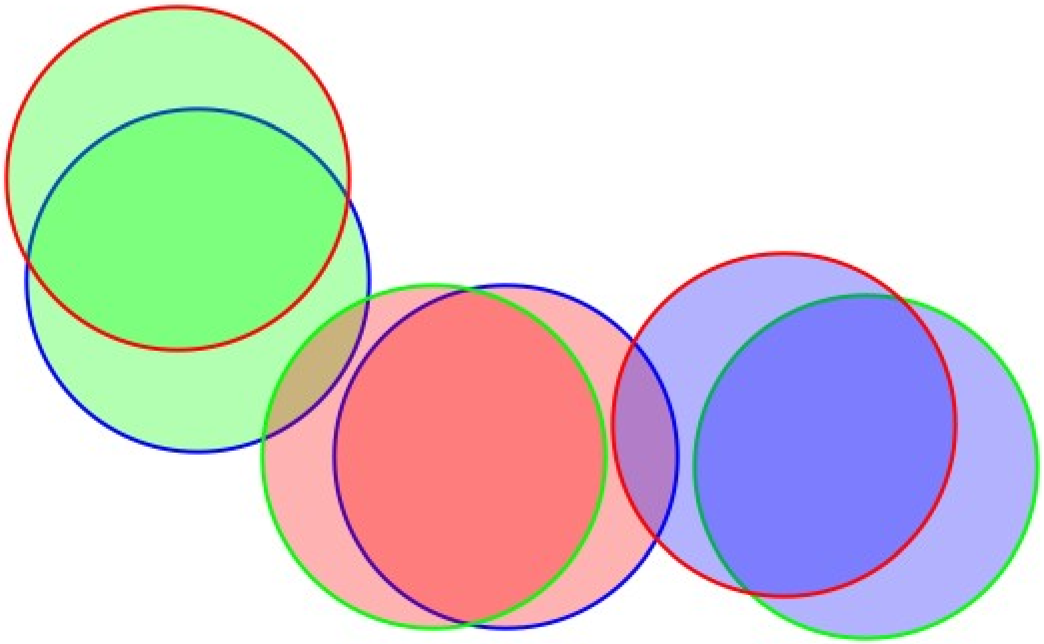

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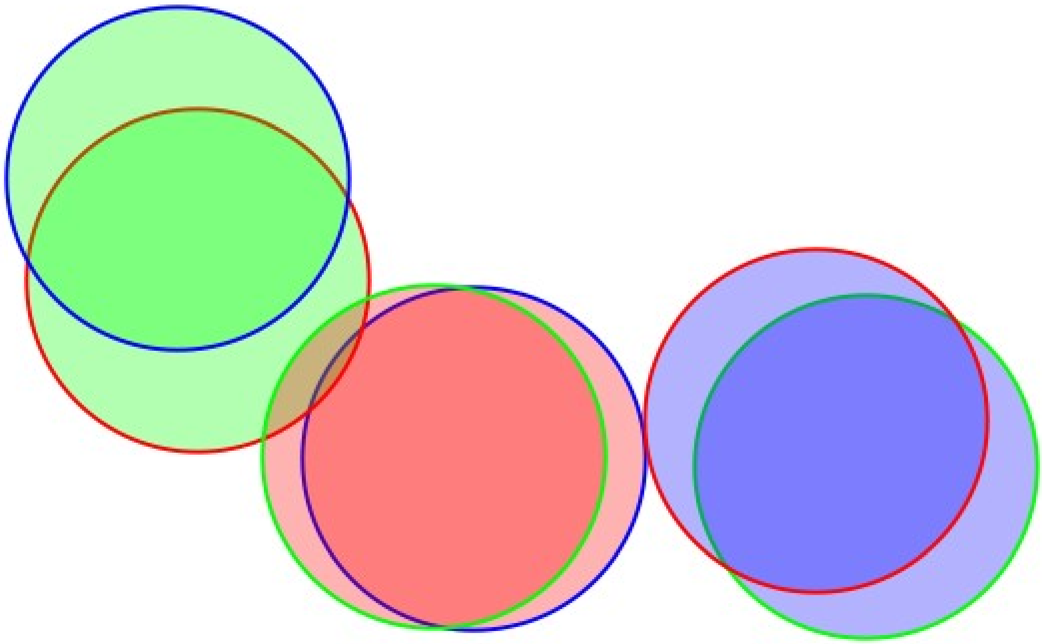
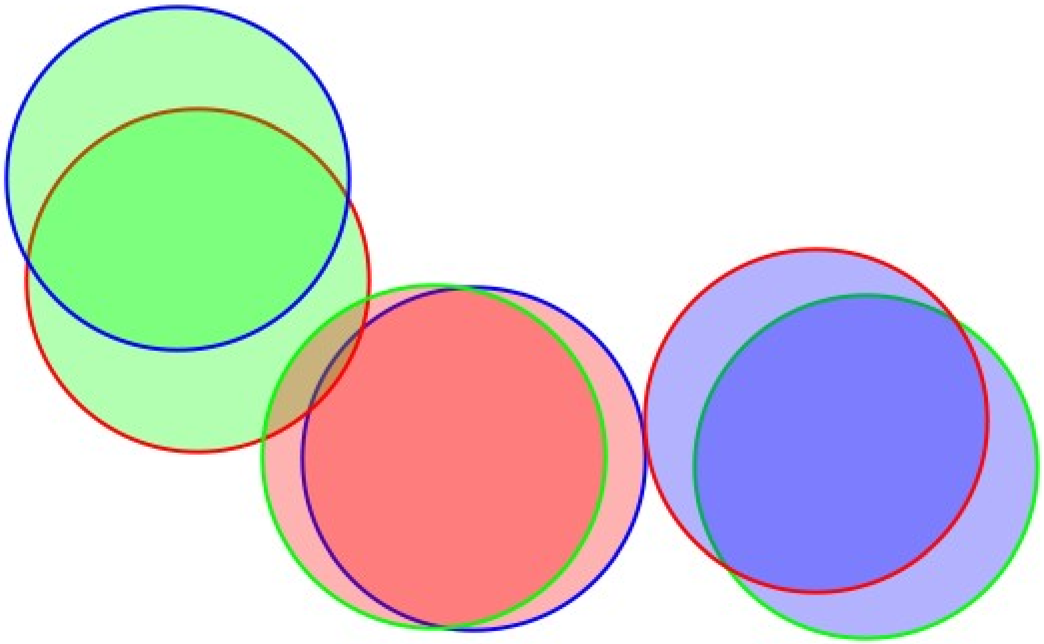
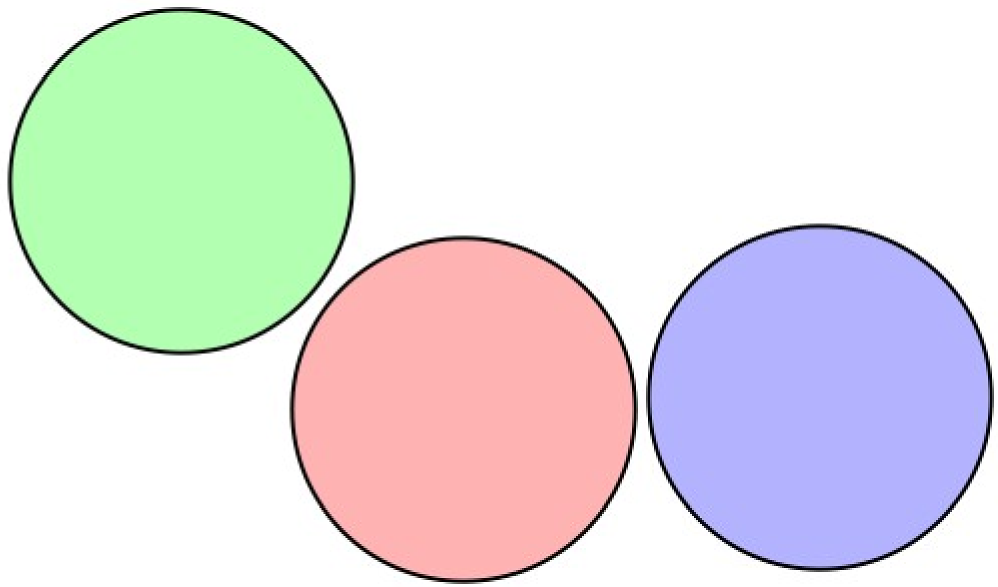


Projection to concurrence constraint



Projection to concurrence constraint



Divide and Concur scheme

A	B
No overlaps between designated replicas	All replicas of a particular figure concur
“divided” packing constraints	“concurrence” constraint

What can we do with projections?

- alternating projections:

$$x_i' = P_A(x_i); \quad x_{i+1} = P_B(x_i')$$

- Douglas-Rachford iteration (a/k/a difference map):

$$x_{i+1} = x_i + P_B(2P_A(x_i) - x_i) - P_A(x_i)$$

Brief (incomplete) history of

$$x_{i+1} = x_i + P_B(2P_A(x_i) - x_i) - P_A(x_i)$$

J. Douglas and H. H. Rachford, *On the numerical solution of heat conduction problems in two or three space variables*, Trans. Am. Math. Soc. 82 (1956), 421–439.

splitting scheme for numerical PDE solutions

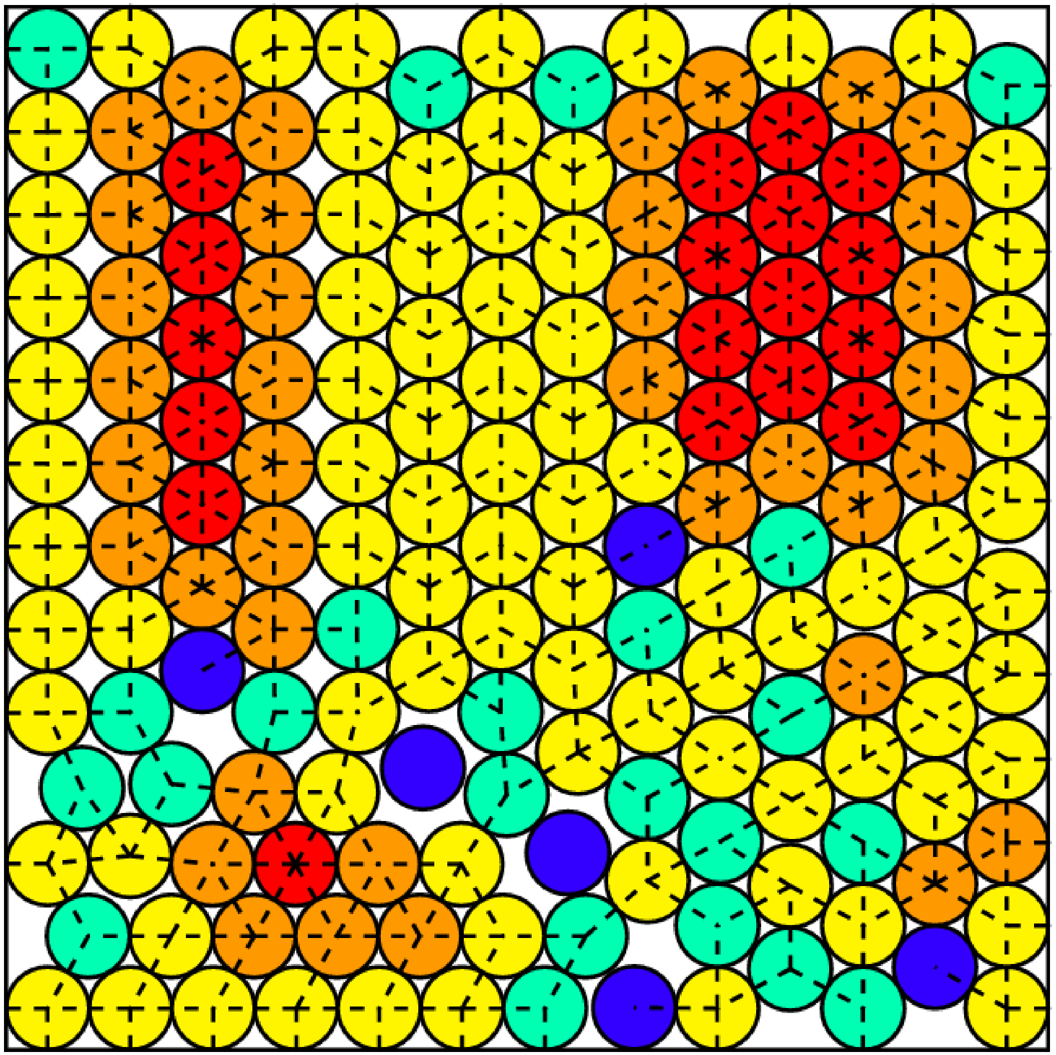
J.R. Fienup, Phase retrieval algorithms: a comparison, Applied Optics 21 (1982), 2758-2769.

rediscovery, control theory motivation, phase retrieval

V. Elser, I. Rankenburg, and P. Thibault, Searching with iterated maps, PNAS 104, (2007), 418-423.

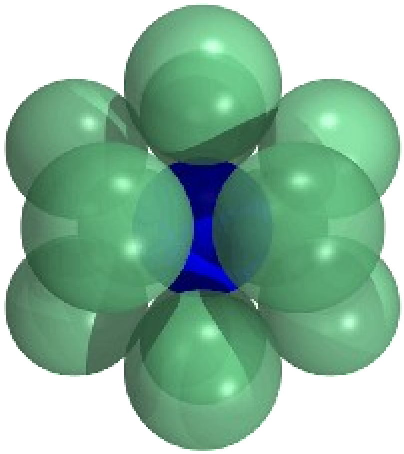
generalized form, applied to hard/frustrated problems:
spin glass, SAT, protein folding, Latin squares, etc.

Finite packing problems

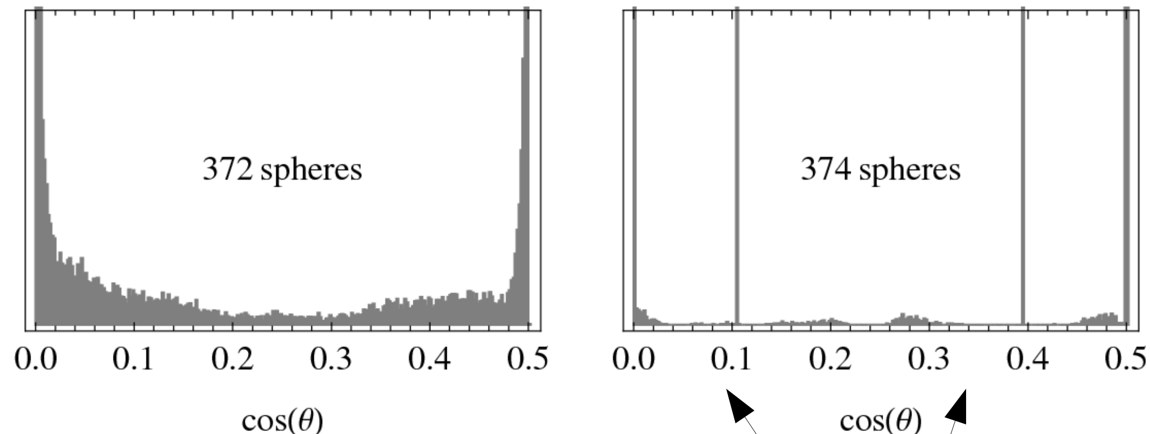


Gravel & Elser, Phys. Rev. E (2008)

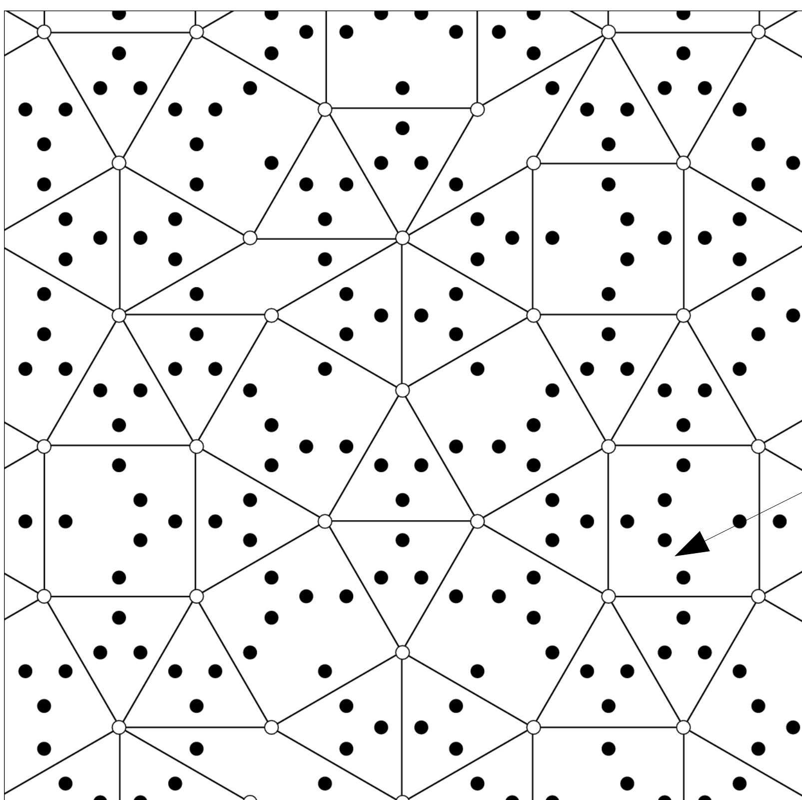
Finite packing problems



The kissing number problem in 10D



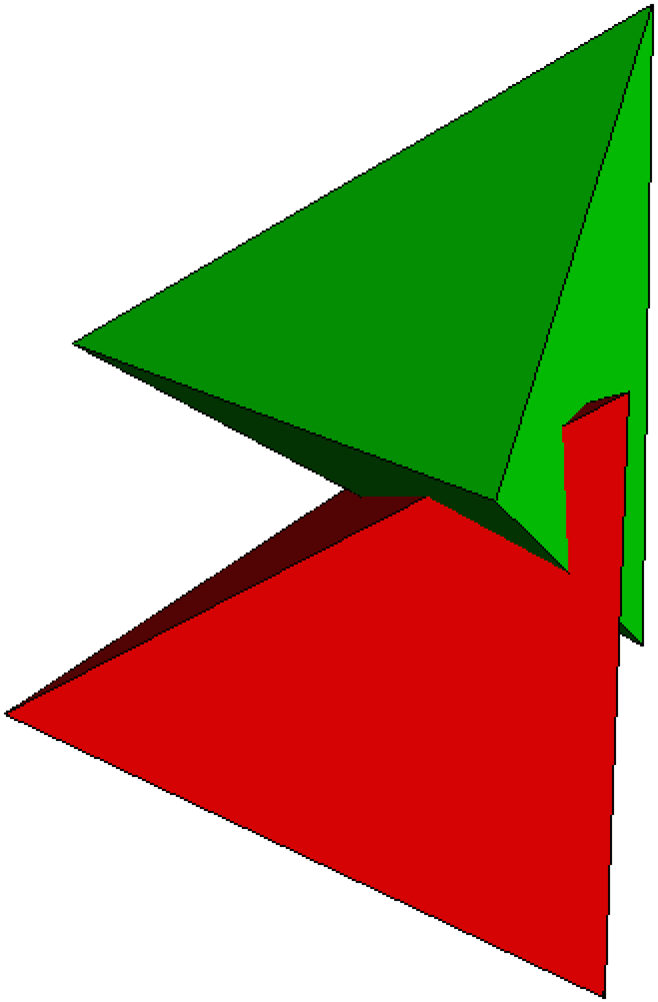
$$(3 \pm \sqrt{3})/12$$



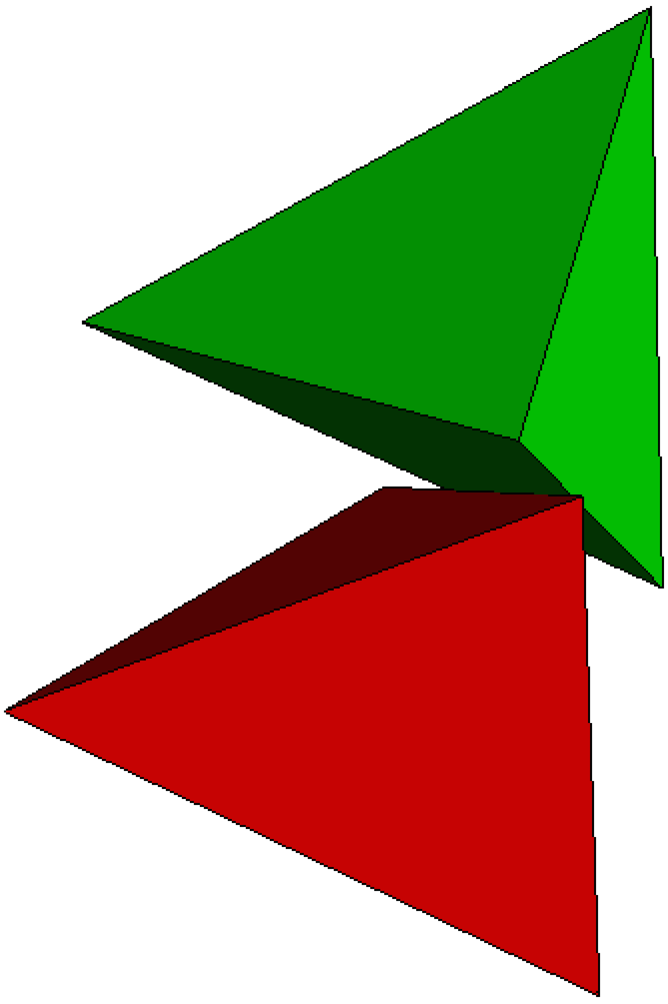
$$A_2 \oplus A_2 \oplus D_4$$

Elser & Gravel, Disc. Compu. Geom. (2010)

Generalization to non-spherical Particles



Generalization to non-spherical Particles



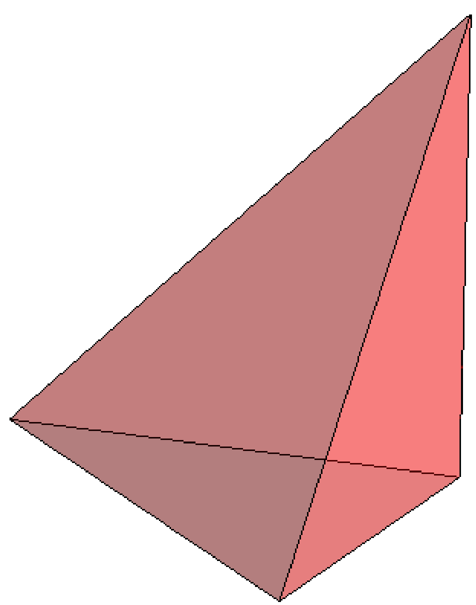
Generalization to non-spherical Particles

A

"divided" packing constraints (rigidity relaxed)

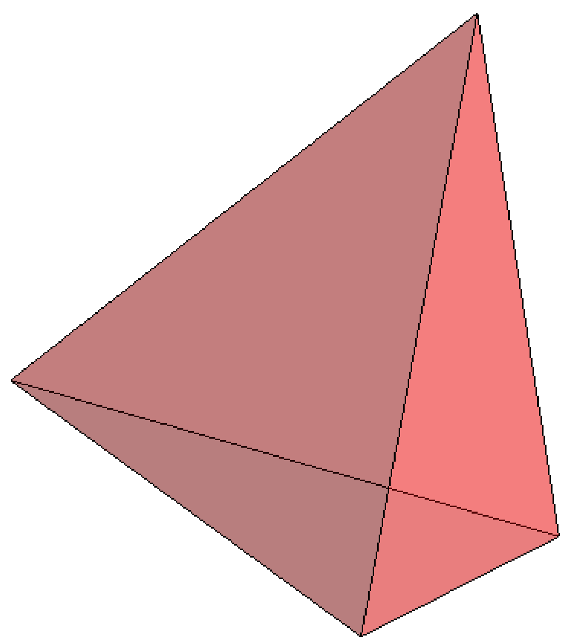
B

"concurrence" + rigidity constraints



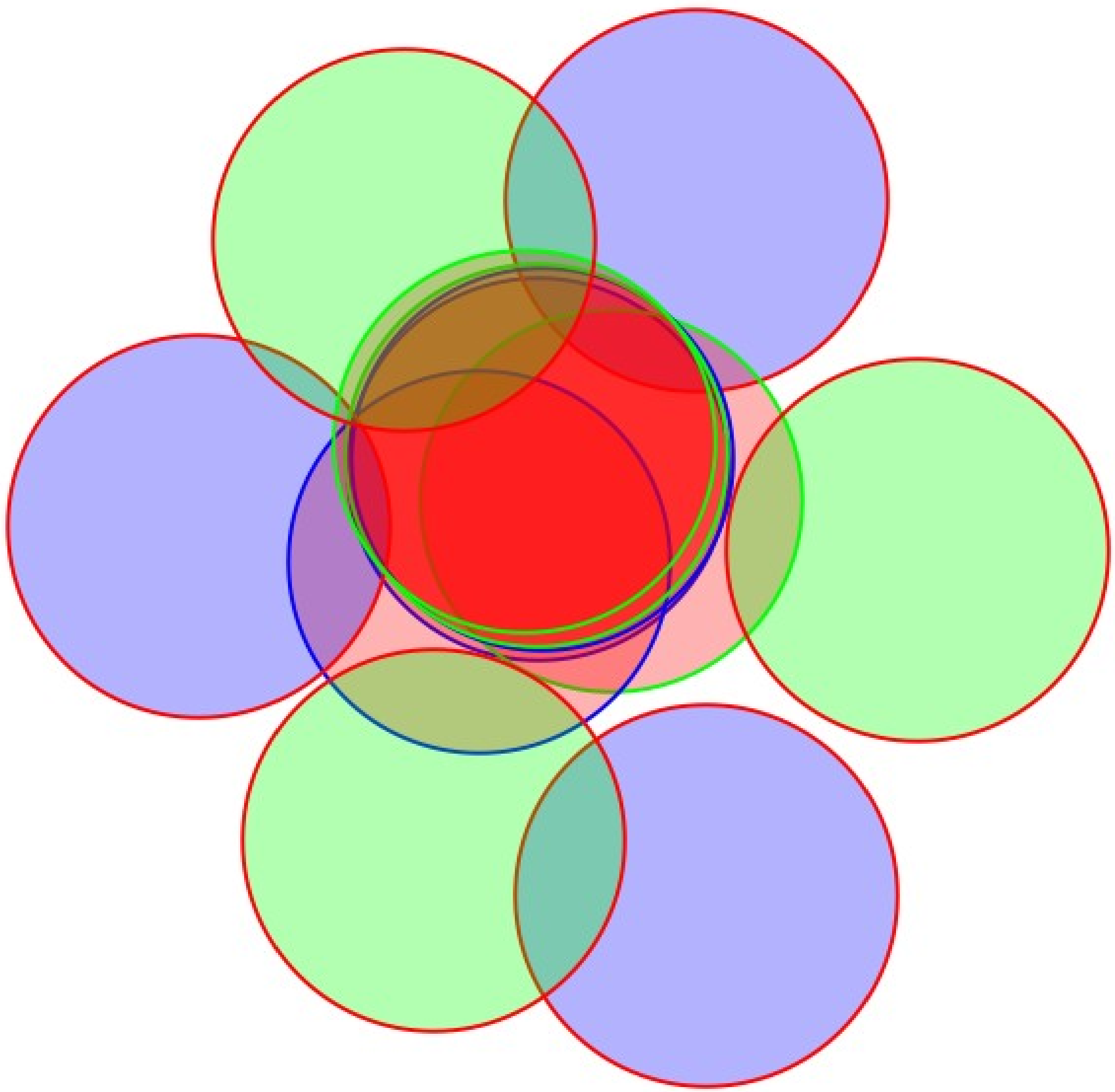
Generalization to non-spherical Particles

A	B
"divided" packing constraints (rigidity relaxed)	"concurrence" + rigidity constraints



Generalization to periodic packings

replicas → replicas + periodic images



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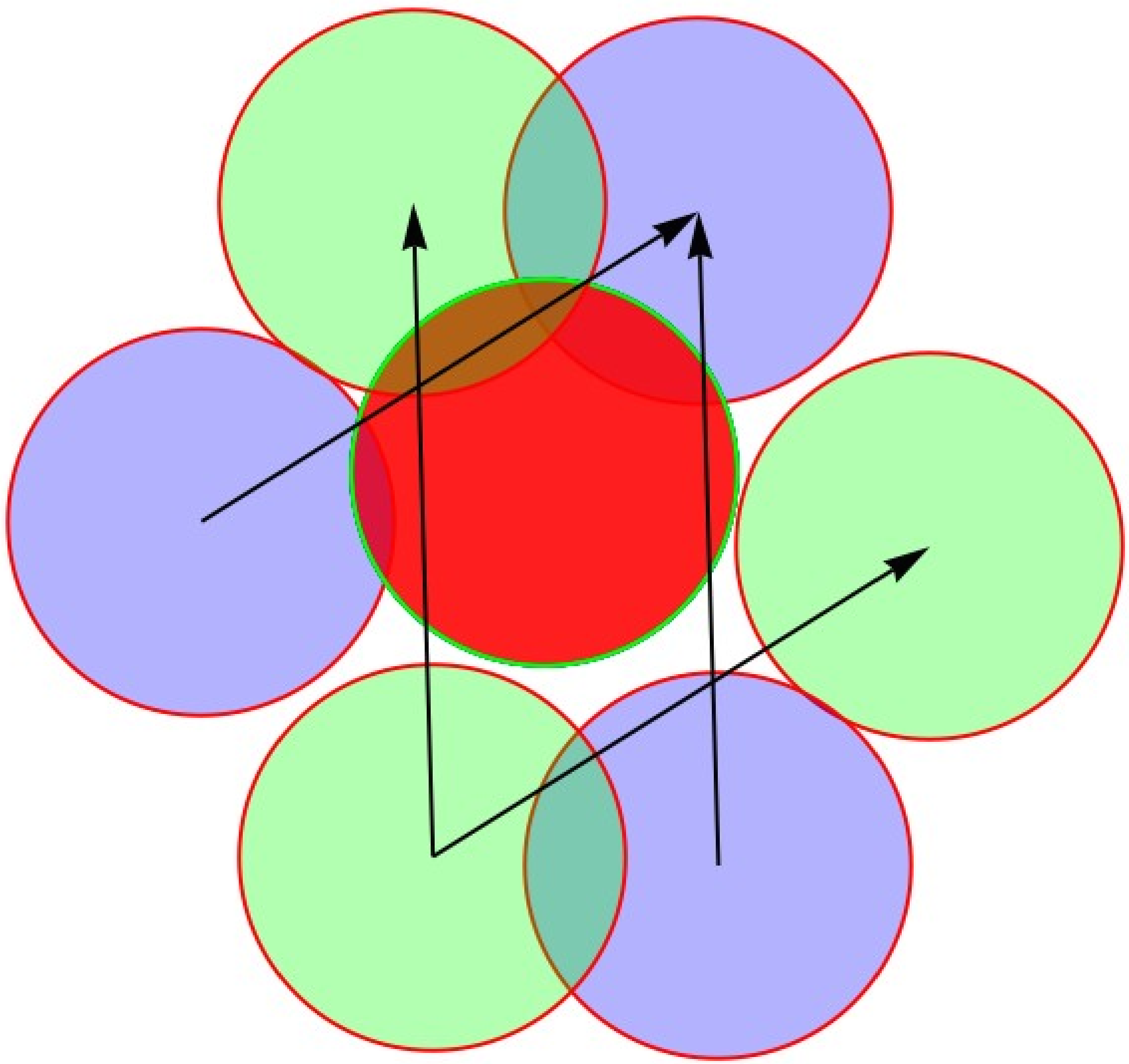
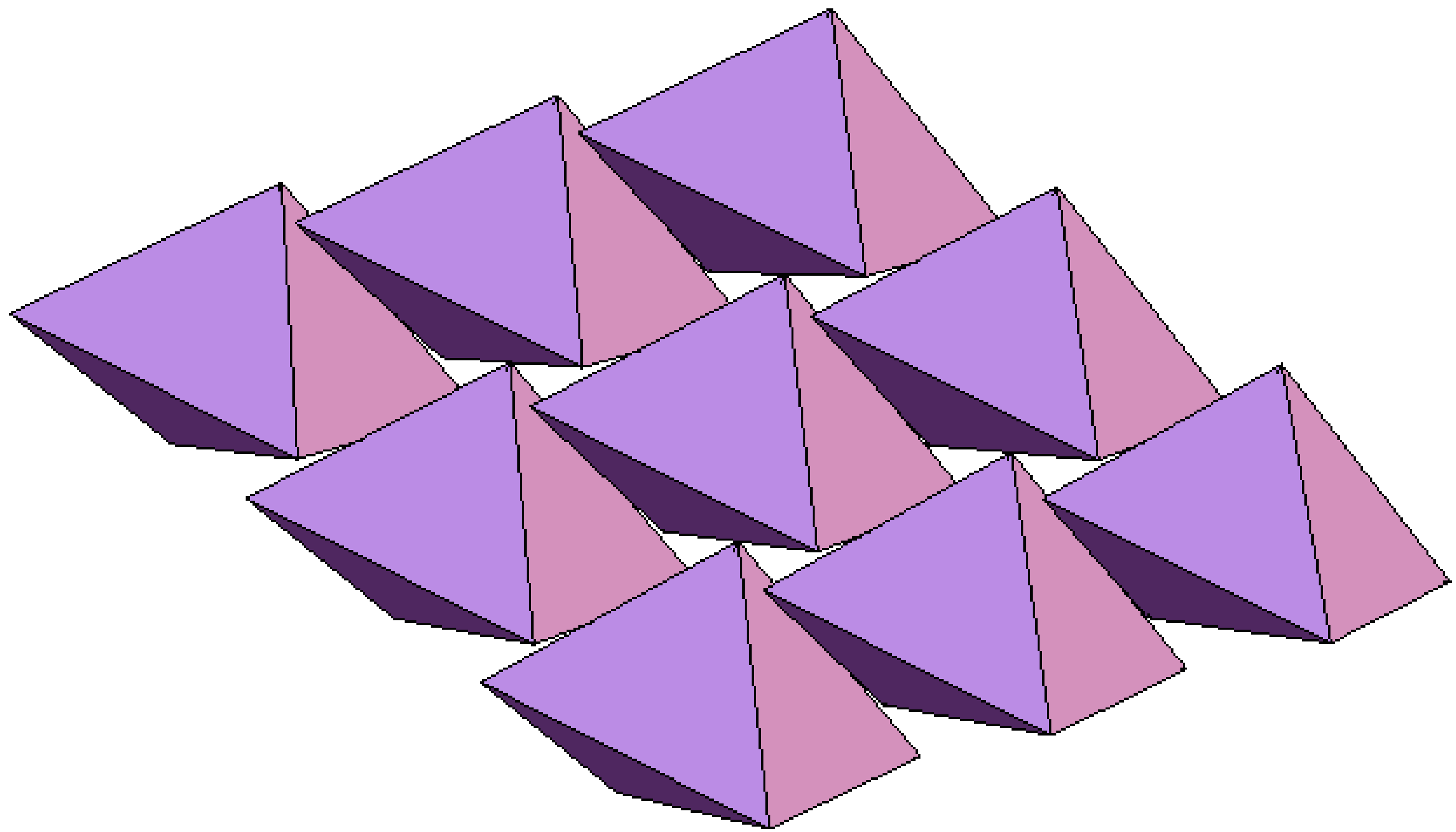


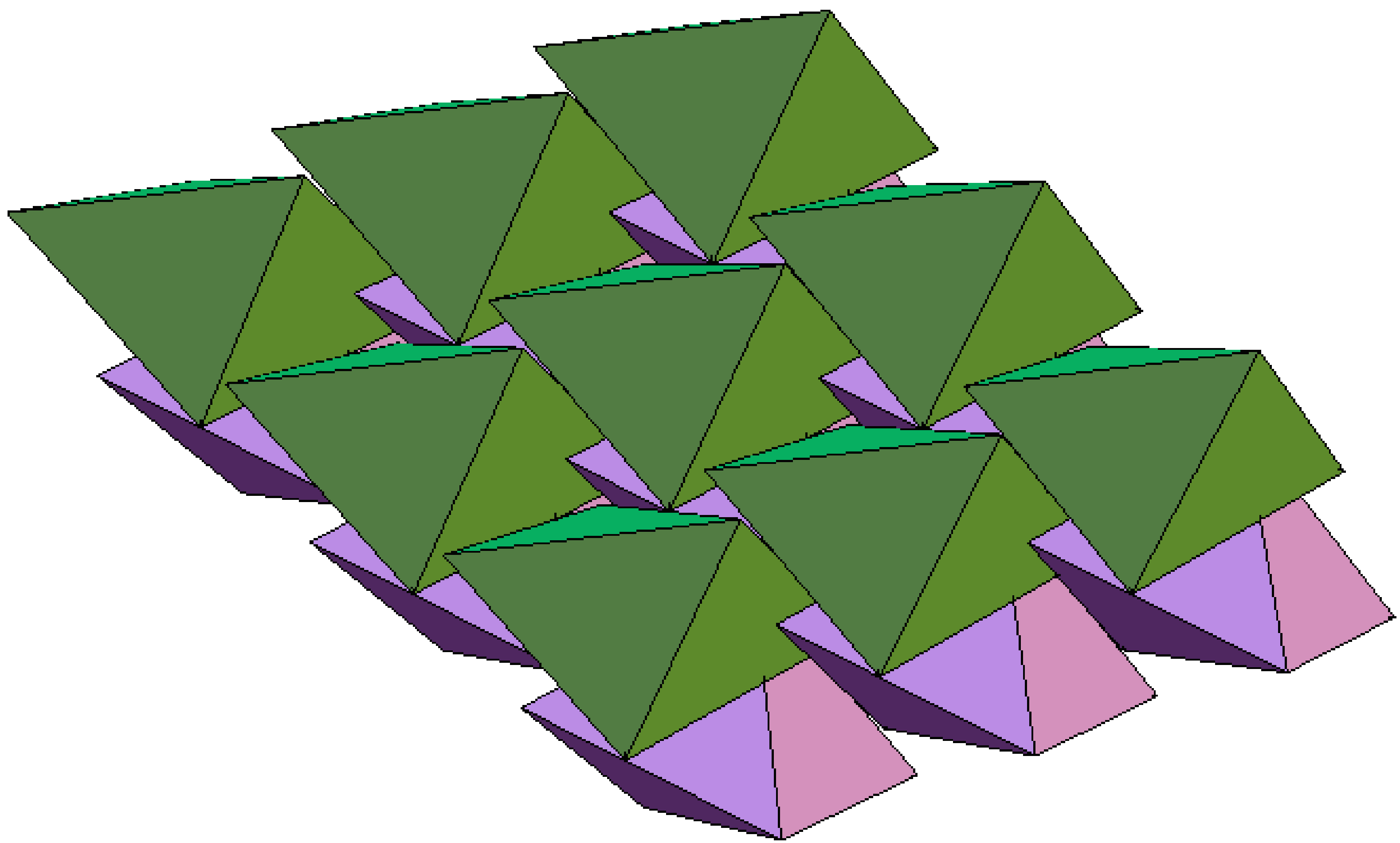
Regular tetrahedron packing

(Stay tuned for next talk)



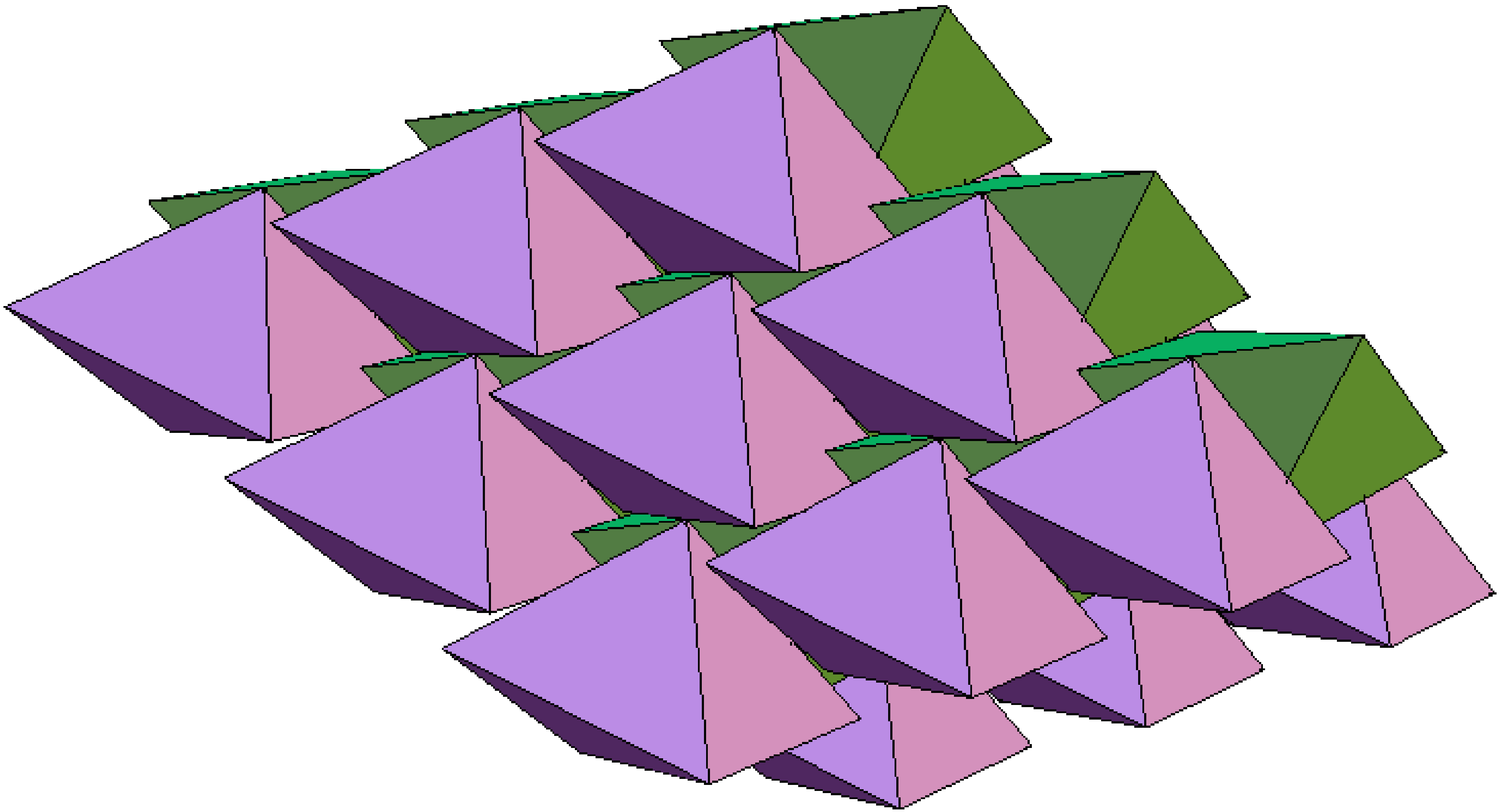
Kallus, Elser, & Gravel, Disc. Compu. Geom. (2010)

Regular tetrahedron packing



Kallus, Elser, & Gravel, Disc. Compu. Geom. (2010)

Regular tetrahedron packing



Kallus, Elser, & Gravel, Disc. Compu. Geom. (2010)

Double-Lattice Packings of Convex Bodies in the Plane

G. Kuperberg[1] and W. Kuperberg[2] Disc. Compu. Geom. (1990)

[1] Department of Mathematics, University of California at Berkeley,
Berkeley, CA 94720, USA

[2] Division of Mathematics, Auburn University, Auburn, AL 36849, USA

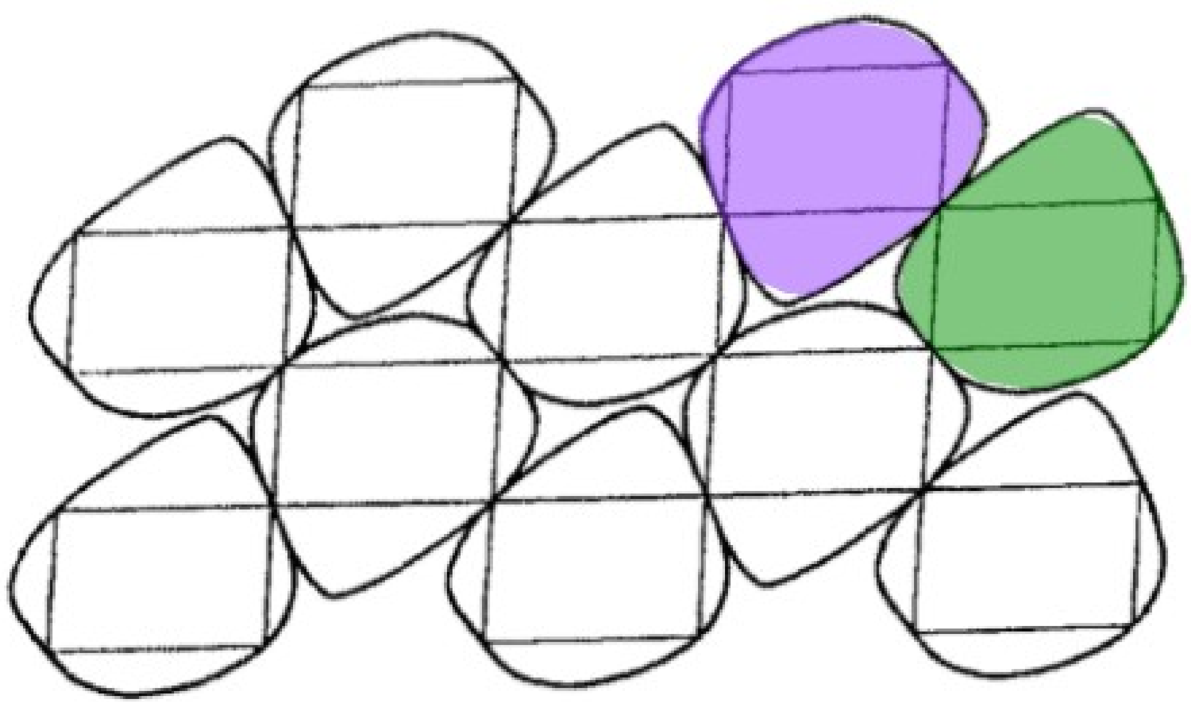
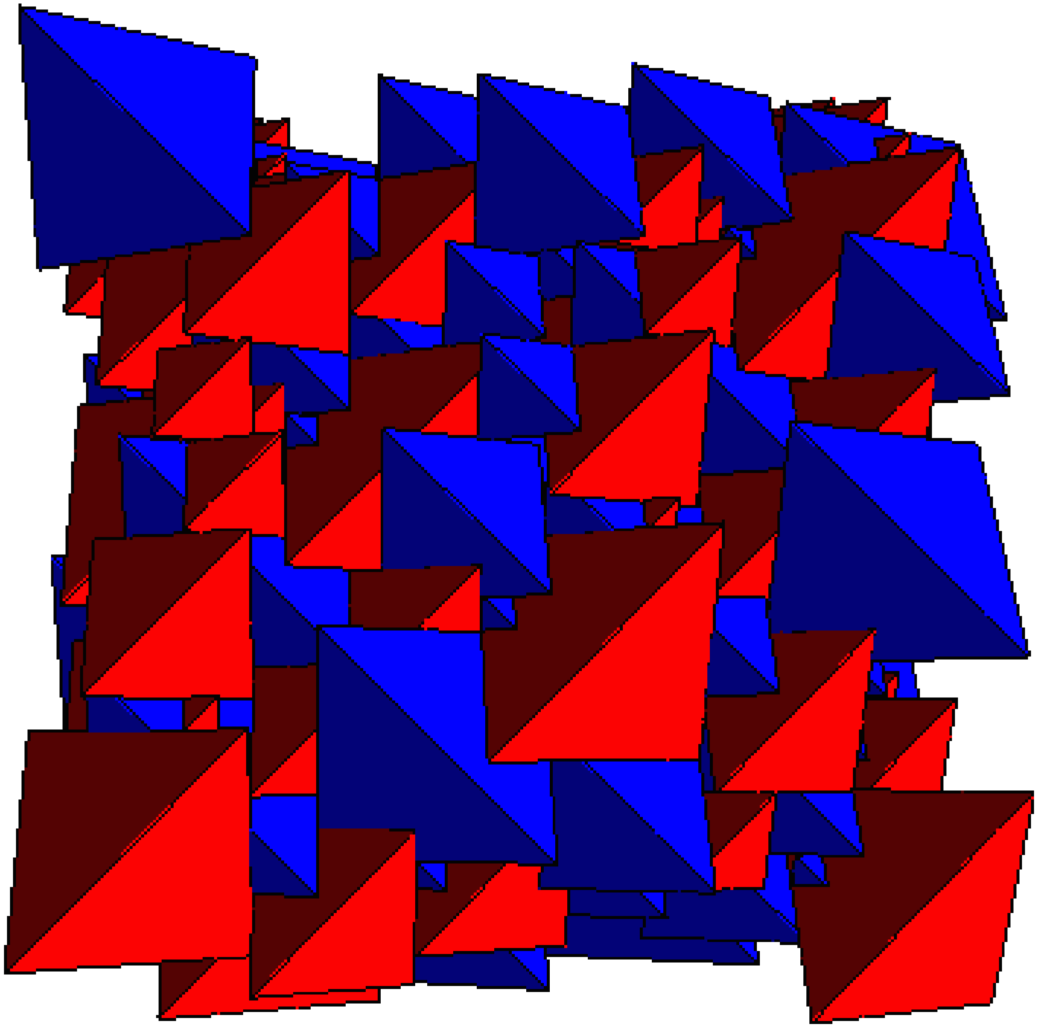


Fig. 4. Double-lattice packing generated by an extensive parallelogram.

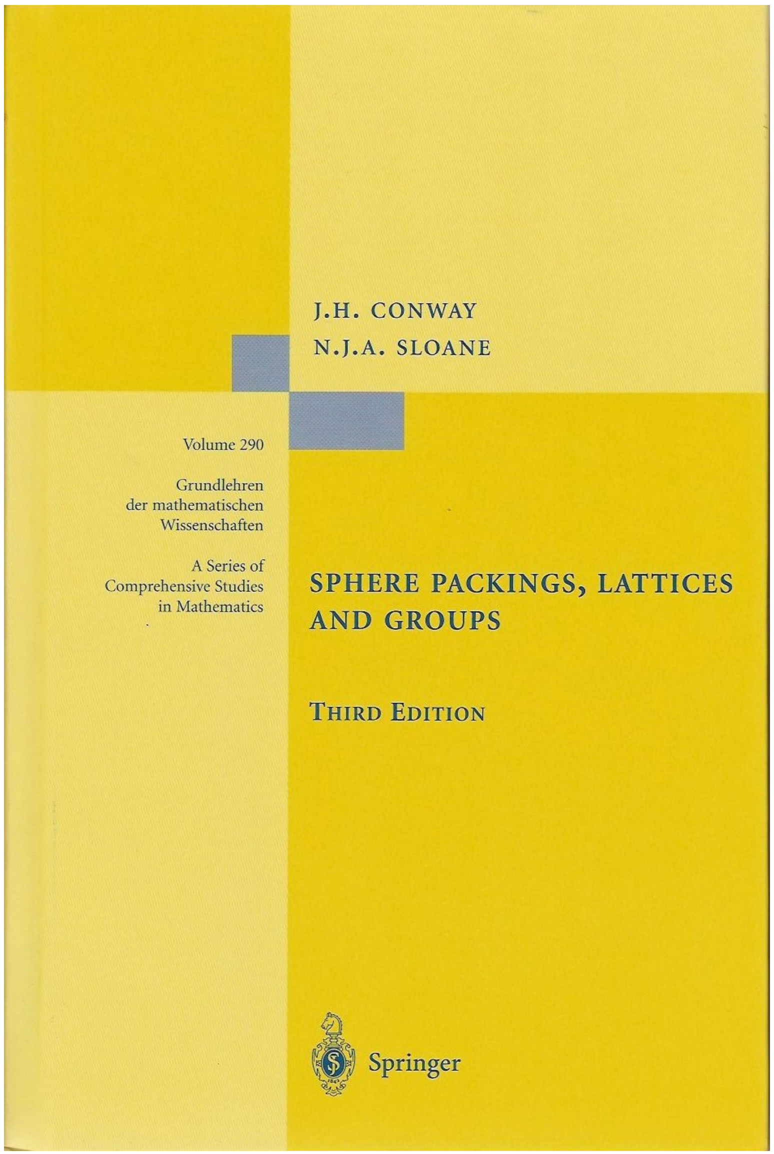
Regular pentatopes!



$\varphi = 128/219 = 0.5845$

Kallus, Elser, & Gravel, arXiv: 1003.3301 (2010)

Sphere packing and kissing in higher dimensions



Densest known lattice packing in *d* dimensions:

d	Λ_{densest}	$\phi^{(L)}_{\text{densest}}$	$\langle N_{\text{iter}} \rangle$
2	A_2	0.90690	42
3	D_3	0.74047	230
4	D_4	0.61685	191
5	D_5	0.46526	308
6	E_6	0.37295	173
7	E_7	0.29530	217
8	E_8	0.25367	99
9	Λ_9	0.14577	161
10	Λ_{10}	0.092021	394
11	K_{11}	0.060432	421
12	K_{12}	0.049454	397
13	K_{13}	0.029208	577
14	Λ_{14}	0.021624	1652

lattice with highest known kissing number in *d* dimensions:

d	Λ_{highest}	$\tau^{(L)}_{\text{highest}}$	$\langle N_{\text{iter}} \rangle$
2	A_2	6	27
3	D_3	12	54
4	D_4	24	132
5	D_5	40	163
6	E_6	72	225
7	E_7	126	597
8	E_8	240	511
9	Λ_9	272	350
10	Λ_{10}	336	438
11	Λ_{11}	438	549

Kallus, Elser, & Gravel, arXiv: 1003.3301 (2010)

Tetrahedron packing upper bound

Optimization challenge:

1. Prove $\varphi \leq 1 - \varepsilon$, where $\varepsilon > 0$
2. Maximize ε

Tetrahedron packing upper bound

Optimization challenge:

1. Prove $\varphi \leq 1 - \varepsilon$, where $\varepsilon > 0$
2. ~~Maximize ε~~
2'. Minimize length of proof

Solution: $\varepsilon = 5.01... \times 10^{-25}$ (15 pages)

Gravel, Elser, & Kallus, preprint