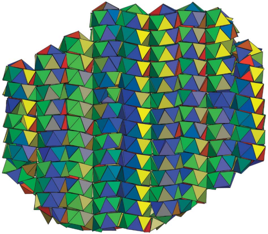
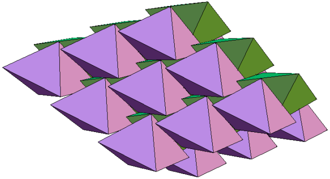
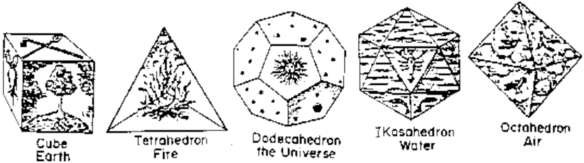
Packing problems: complex structure from simple interactions

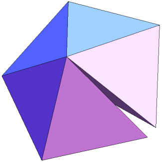




Yoav Kallus
Santa Fe Institute
January 5, 2017

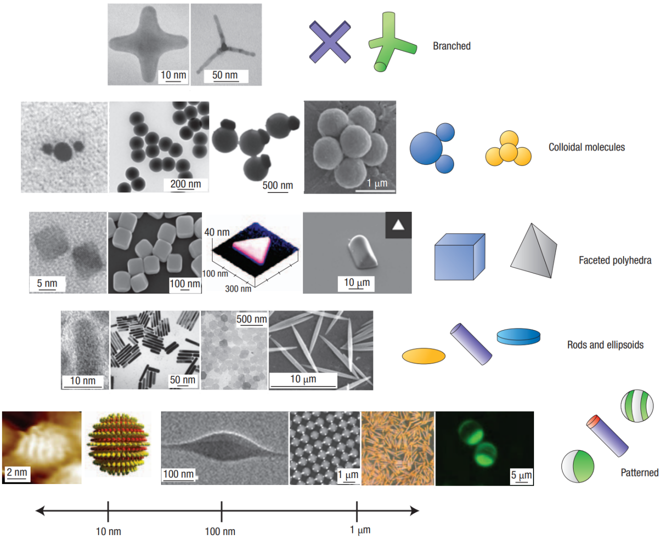
The long history of packing problems





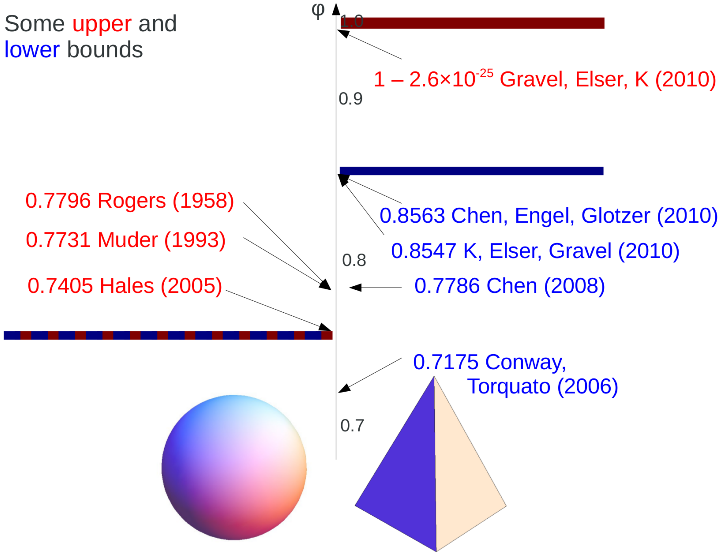
"In general, the attempt to give a shape to each of the simple bodies is unsound, for the reason, first, that they will not succeed in filling the whole. It is agreed that there are only three plane figures which can fill a space, the triangle, the square, and the hexagon, and only two solids, the pyramid [tetrahedron] and the cube."
– Aristotle. *On the Heavens*, volume III

Building blocks by design

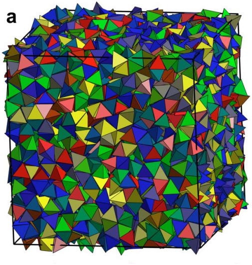


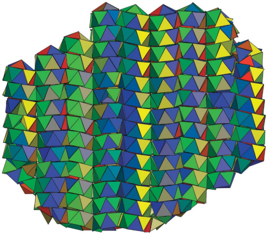
Glotzer and Solomon, Nature Materials 2007

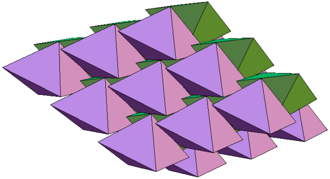
Packing spheres vs. tetrahedra



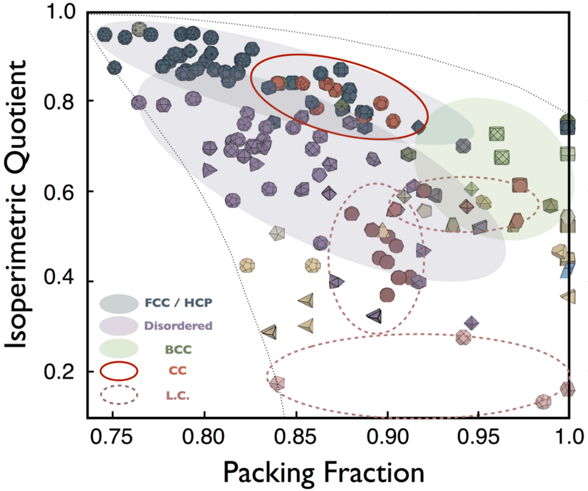
Emergent structure in tetrahedron packing







Packing convex shapes



Ulam's conjecture: balls are worst among convex shapes

Worst packing shapes

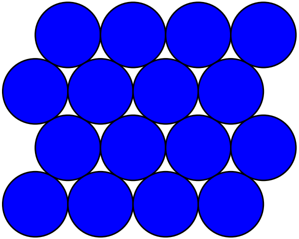
Best packing shapes are trivial

Worst packing shapes

Best packing shapes are trivial

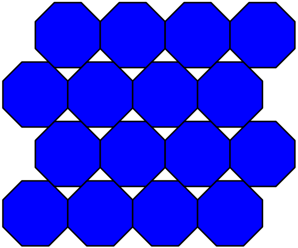
Worst shape is a more interesting question

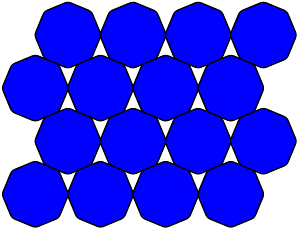
In 2D disks are not worst



$\phi = 0.9069$

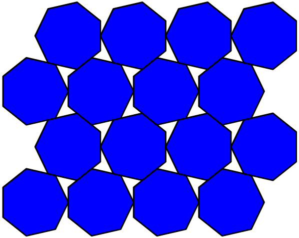
$\phi = 0.9062$



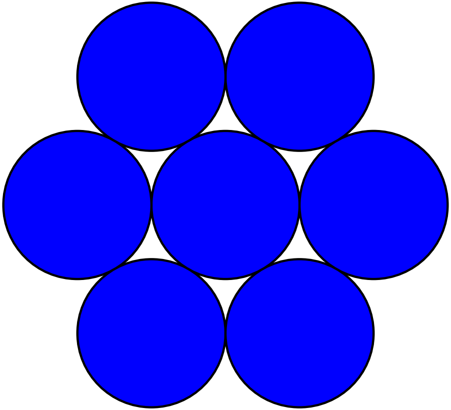


$\phi = 0.9024$

$\phi =$ 0.8926(?)



Why can we improve over circles?



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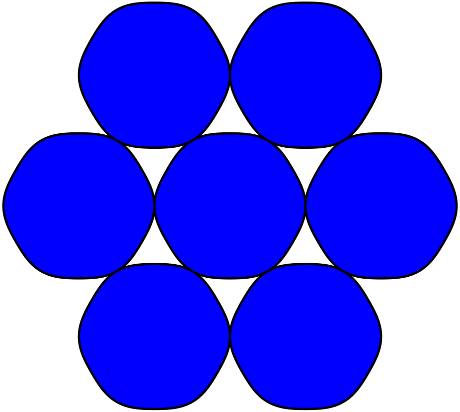

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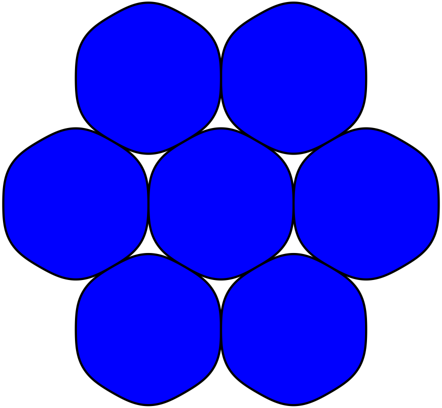

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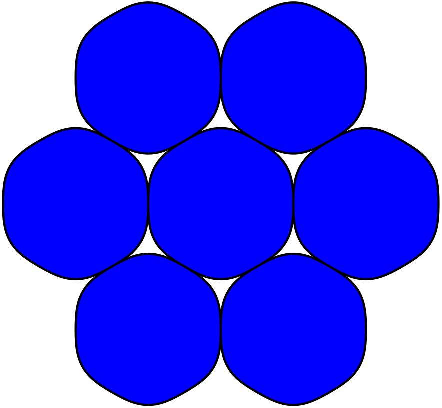


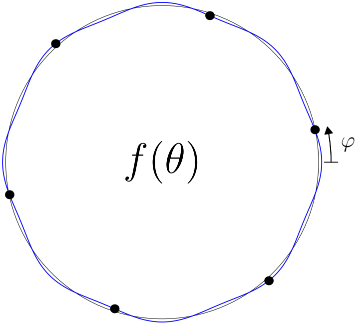
To first order:

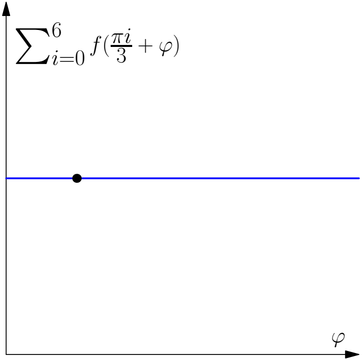
Δ(vol. per particle) $\propto$ avg. deformation in contact dirs.

Δ(vol. of particle) $\propto$ avg. deformation in all dirs.

Can only break even, and make up in higher orders

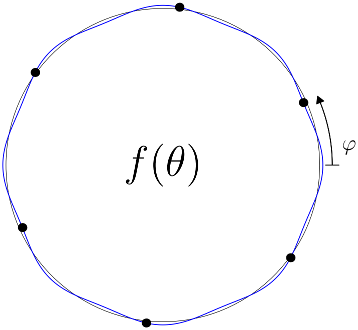
Why can we improve over circles?

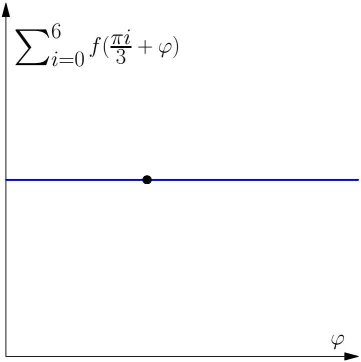




$$f(\theta) = 1 + \epsilon cos(8\theta)$$

Why can we improve over circles?





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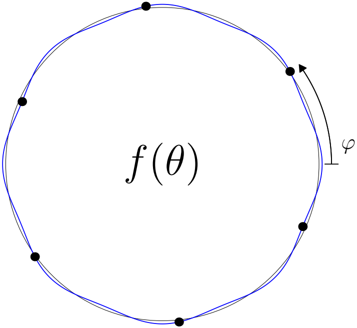

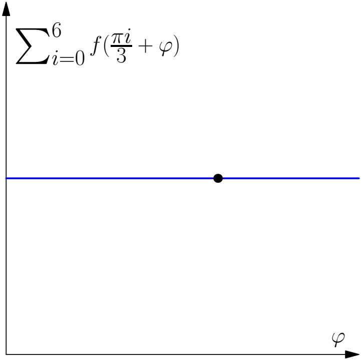

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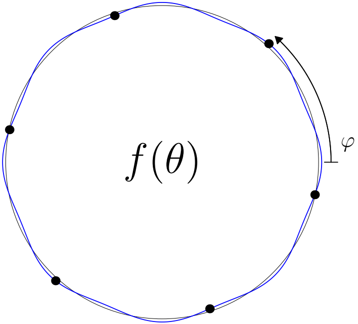

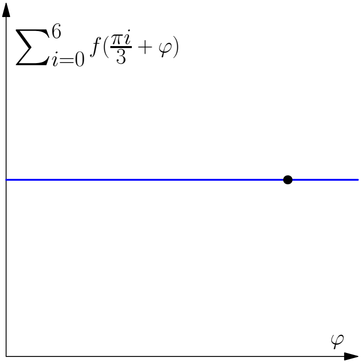

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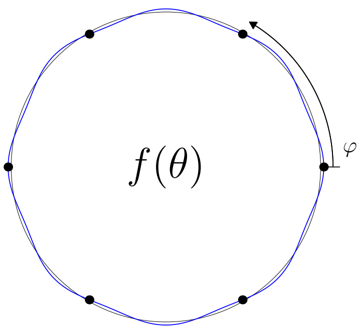

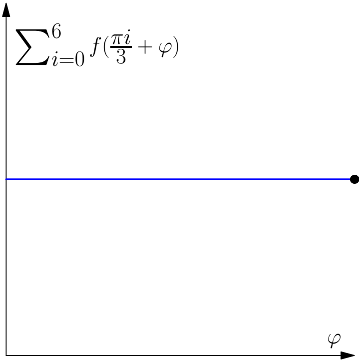

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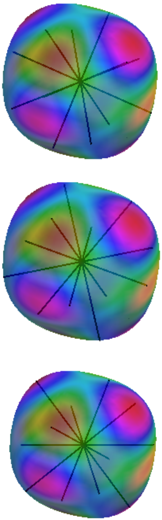
Why can we not improve over spheres?



Let $\mathbf{x}_i$, $i = 1, \ldots, 12$, be the twelve contact points on the sphere in the f.c.c. packing.

Lemma

Let f be an even function $S^2 \to \mathbb{R}$. $\sum_{i=1}^{12} f(R\mathbf{x}_i)$ is independent of $R \in SO(3)$ if and only if the expansion of $f(\mathbf{x})$ in spherical harmonics terminates at $l = 2$.

K, Adv Math 2014

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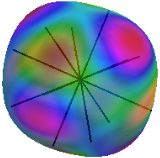

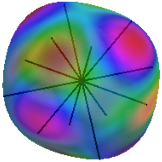

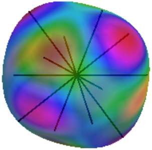

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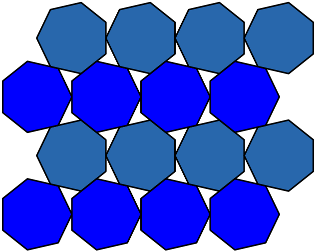
Theorem (K)

The sphere is a local minimum of ϕ, the packing density, among convex, centrally symmetric bodies.

K, Adv Math 2014

Heptagons are locally worst packing (?)



0.8926(?)

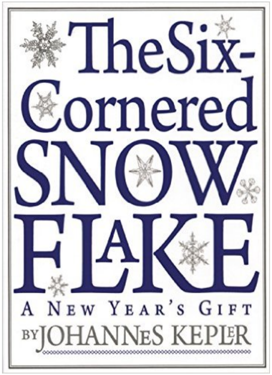
Theorem (K)

Any convex body sufficiently close to the regular heptagon can be packed at a filling fraction at least that of the "double lattice" packing of regular heptagons.

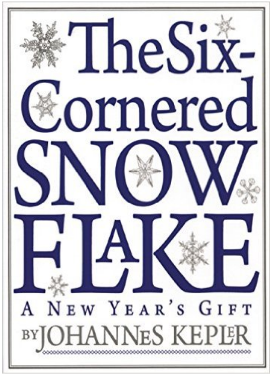
It is not proven, but highly likely, that the "double lattice" packing is the densest packing of regular heptagons.

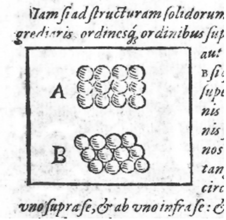
K, Geometry & Topology 2015

A New Year's gift



A New Year's gift





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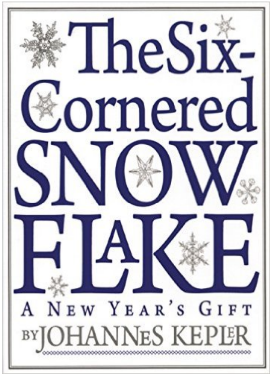
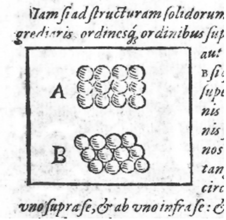
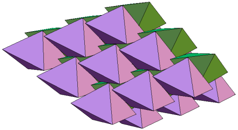
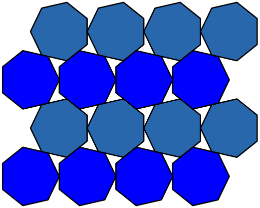
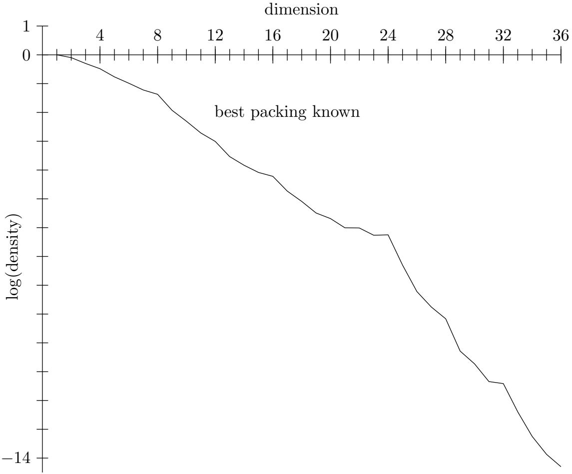




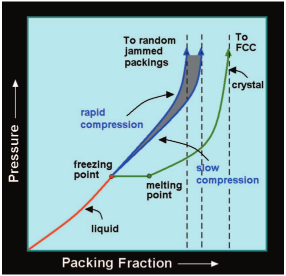
Optimal packing is idiosyncratic

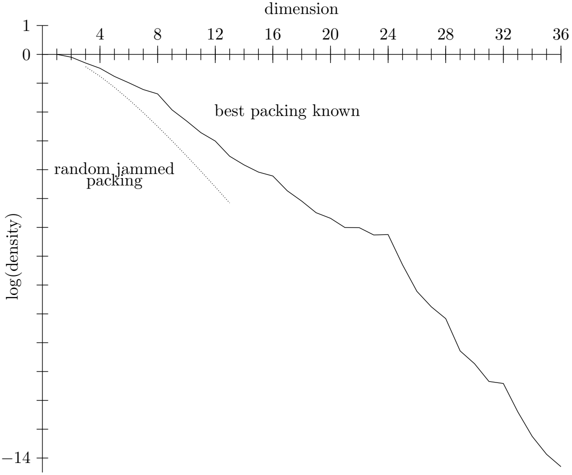




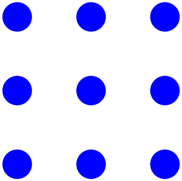


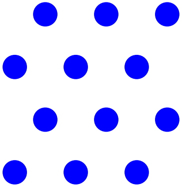
Disordered packing behaves smoothly

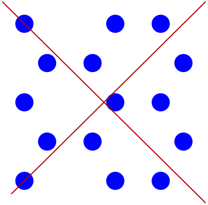


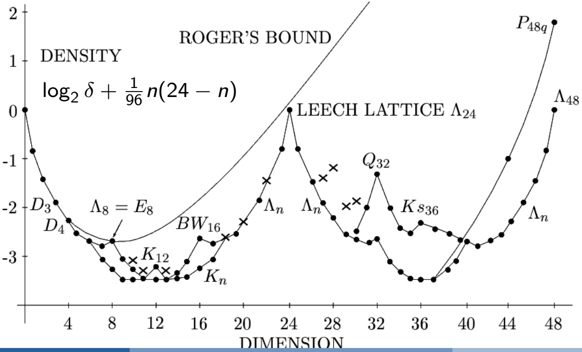


Restricting the sphere packing problem to Bravais lattices

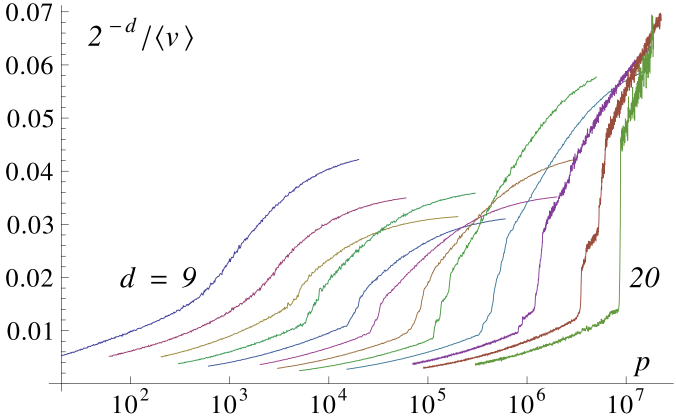








Thermodynamically sampling lattices

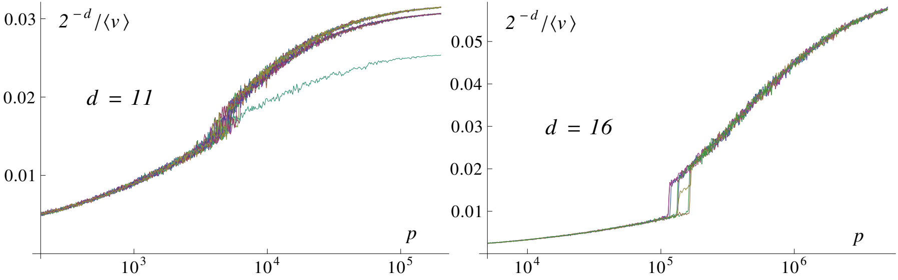


Prob. $\sim \exp(-pV_{\text{unit cell}})$

Densest known lattice recovered in some runs for $d \leq 20$

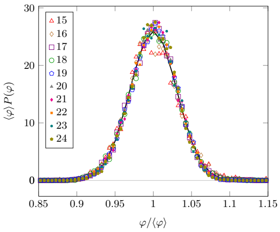
K, Phys. Rev. E 87, 063307 (2013)

Thermodynamically sampling lattices

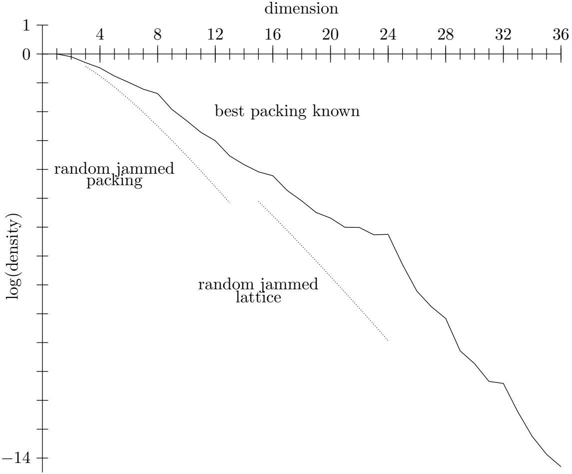


K, Phys. Rev. E 87, 063307 (2013)

Lattice RCP

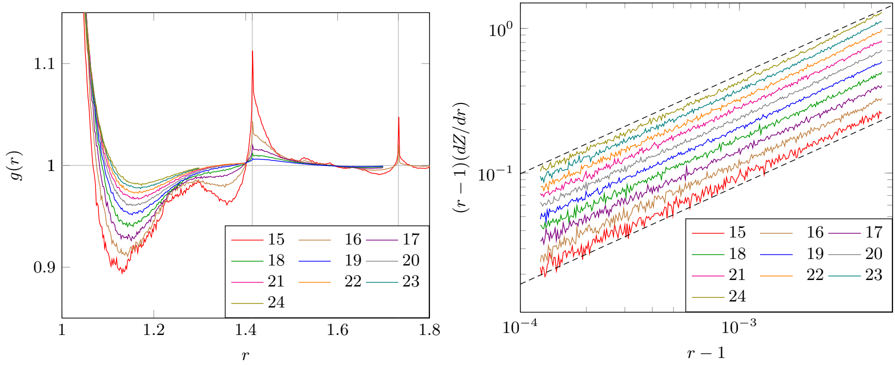


30
20
10
0
15
16
17
18
19
20
21
22
23
24
0.85
0.9
0.95
1
1.05
1.1
1.15
$\langle\varphi\rangle P(\varphi)$
$\varphi/\langle\varphi\rangle$



dimension
1
0
−14
4
8
12
16
20
24
28
32
36
best packing known
random jammed
packing
random jammed
lattice
log(density)

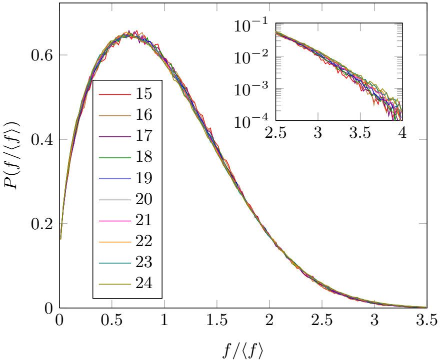
Pair correlations and quasicontacts

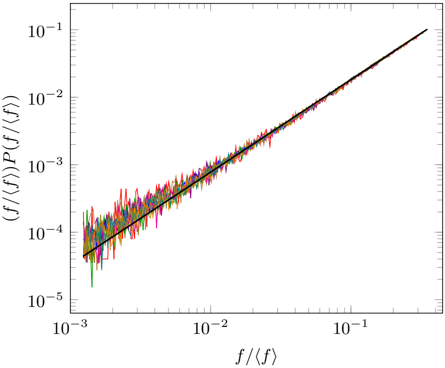


$$g(r) \sim (r-1)^{-\gamma}$$
$$Z(r) \sim d(d+1) + A_d(r-1)^{1-\gamma}$$
$$\gamma = 0.314 \pm 0.004$$

K, Marcotte, & Torquato, Phys. Rev. E 88, 062151 (2013)

Contact force distribution

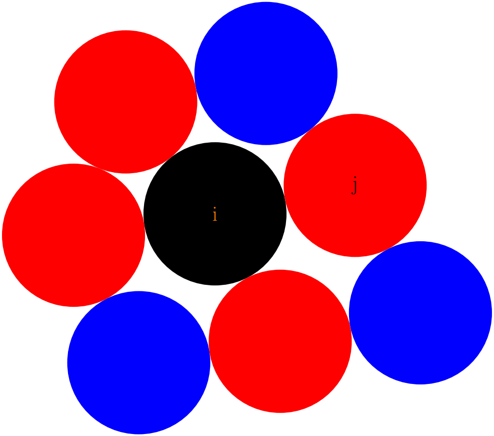




$$P(f) \sim f^{\theta}$$
$$\theta = 0.371 \pm 0.010$$

K, Marcotte, & Torquato, Phys. Rev. E 88, 062151 (2013)

Importance of marginal contacts



Wyart, Phys. Rev. Lett. 109, 125502 (2012)

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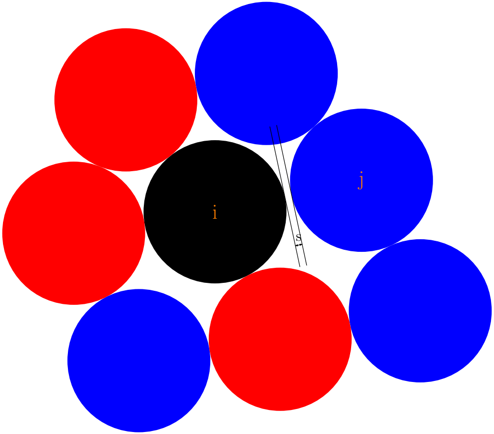

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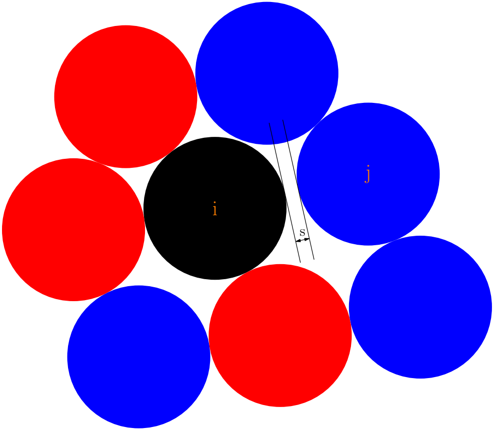

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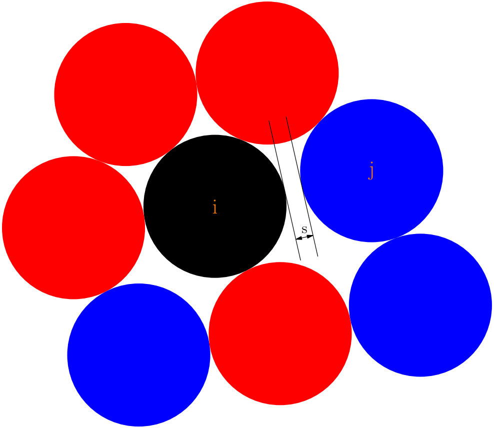

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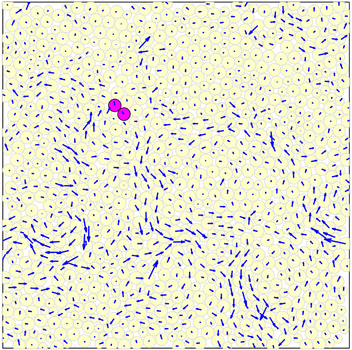
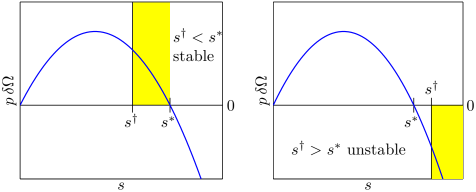
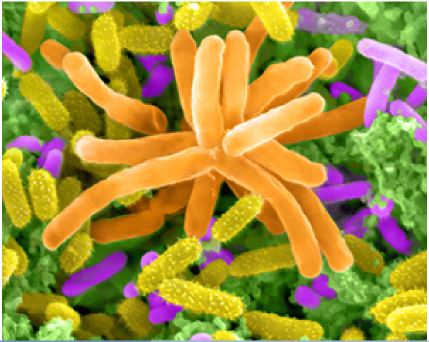


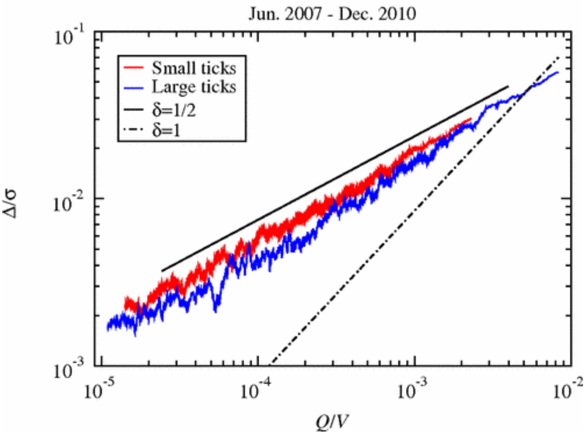


Wyart, Phys. Rev. Lett. 109, 125502 (2012)

Self-organized structure in complex systems



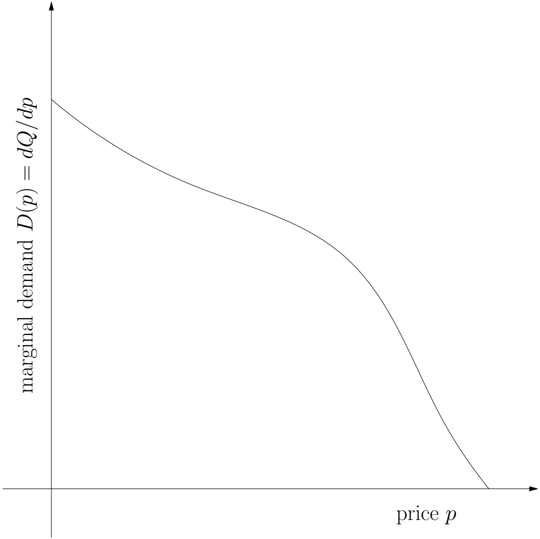
Anomalous price impact



Price impact from 5×10^5 trades on futures market by J.-P. Bouchaud's CFM (Tóth et al., PRX **1**, 021006 (2011)).

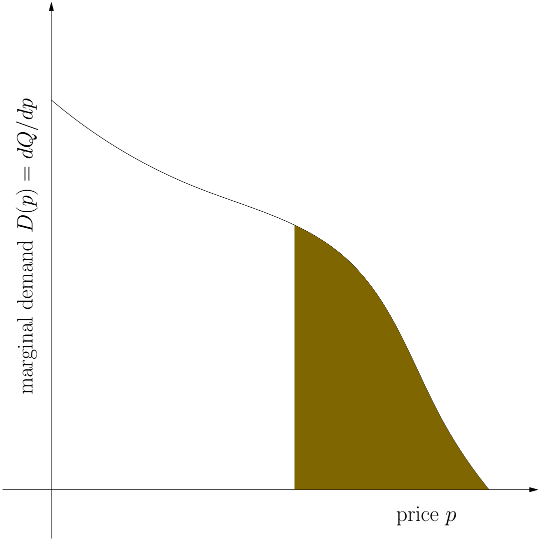
Why anomalous?

Consider a good with a fixed supply. How is its price determined?



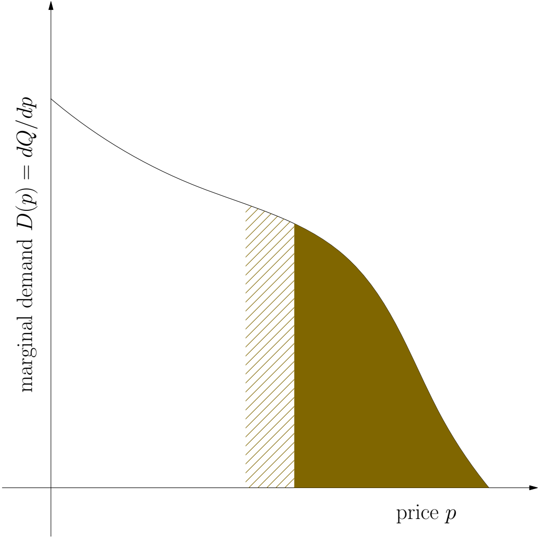
Why anomalous?

Consider a good with a fixed supply. How is its price determined?



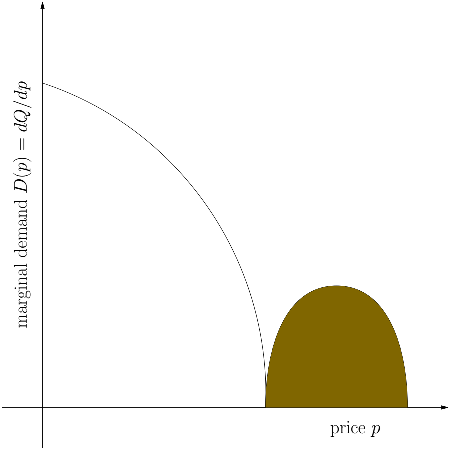
Why anomalous?

When supply changes, how does price change?

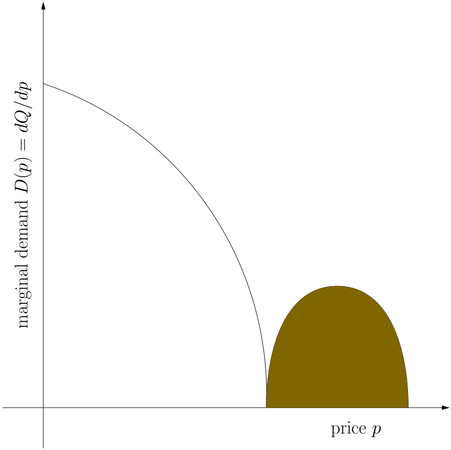


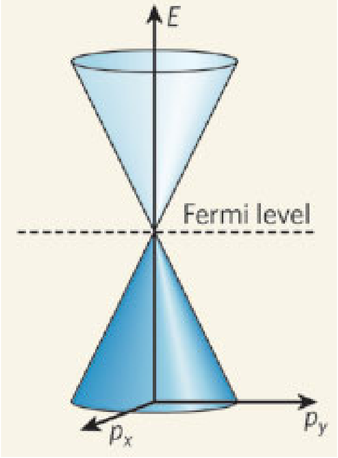
$$\Delta = -DQ + O(Q^2)$$

Anomalous impact means vanishing liquidity

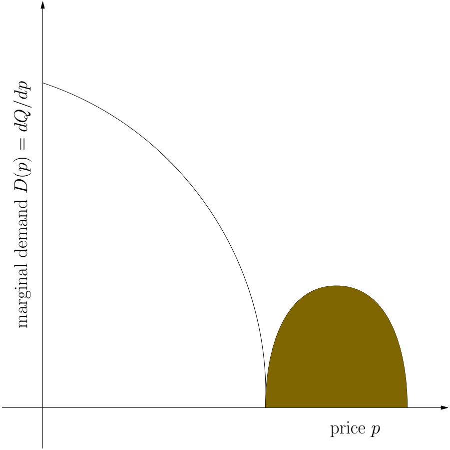


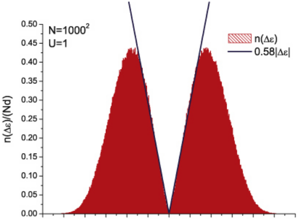
Anomalous impact means vanishing liquidity



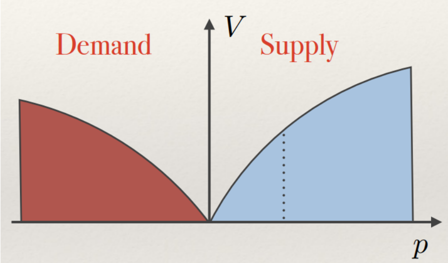


Anomalous impact means vanishing liquidity



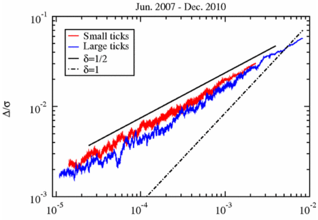


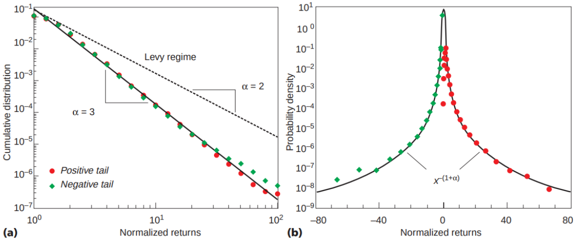
Toy model of trading on a rugged utility landscape



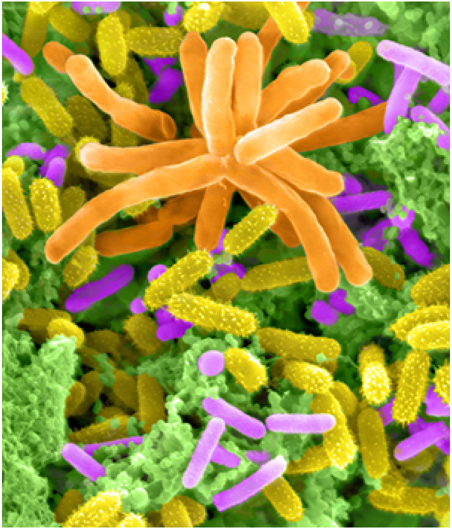
N different goods, M agents with utility function $U_i = U(x_{i,1}, \ldots, x_{i,N})$. Trade good j if $\partial U/\partial x_{i,j} > \partial U/\partial x_{i',j}$, $x_{i',j} > 0$, and $x_{i,j} < x_{\max}$.

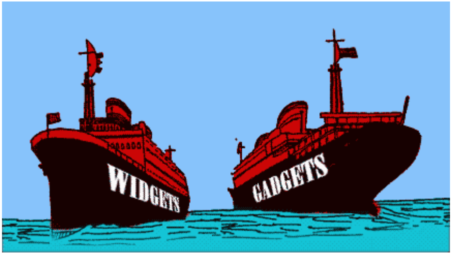
Toy model of trading on a rugged utility landscape





Trading in bacterial communities

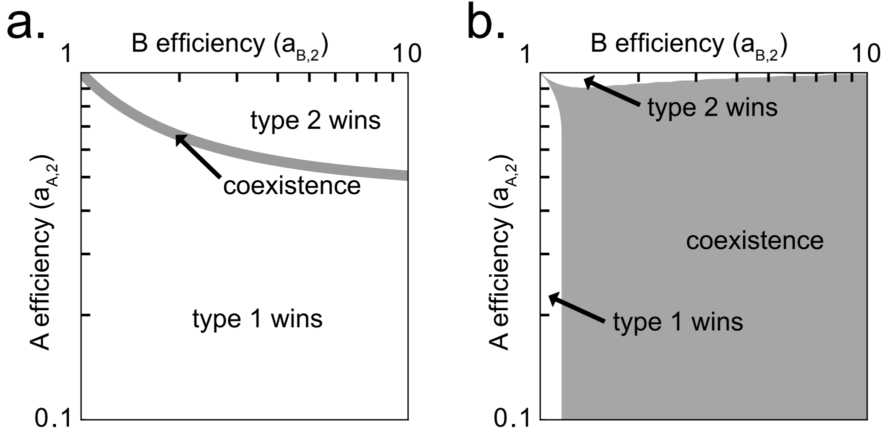




Imporatnt differences: population dynamics, goods leak, no direct exchange.

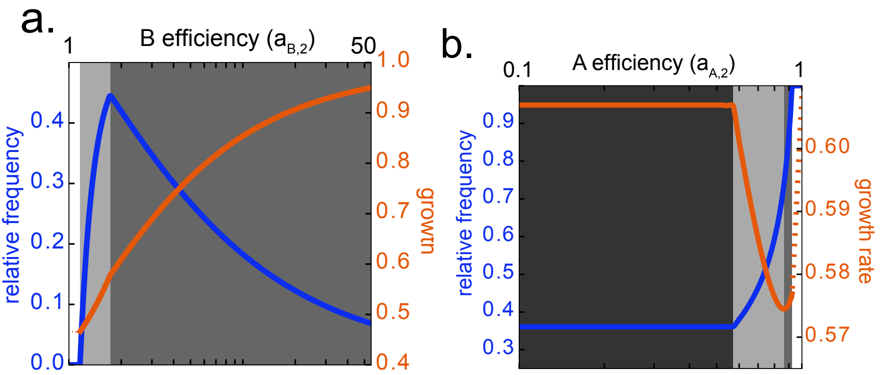
K, Miller, & Libby, arXiv:1612.03125

Spontaneous emergence of beneficial trade



K, Miller, & Libby, arXiv:1612.03125

The curse of increased efficiency



K, Miller, & Libby, arXiv:1612.03125