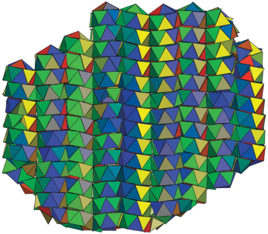
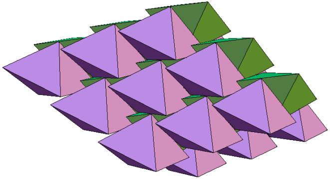
Packing problems: complex structure from simple interactions



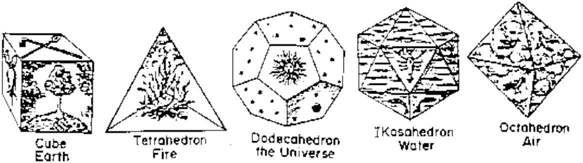


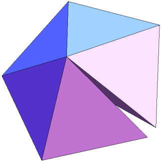
Yoav Kallus

Santa Fe Institute

February, 2017

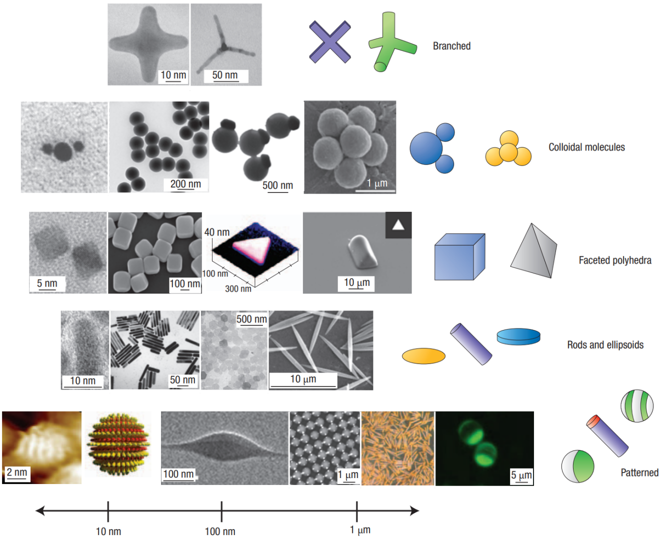
The long history of packing problems





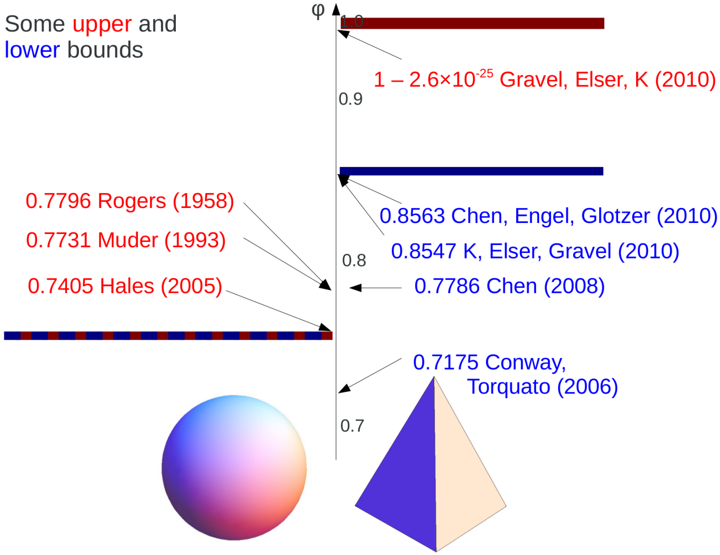
"In general, the attempt to give a shape to each of the simple bodies is unsound, for the reason, first, that they will not succeed in filling the whole. It is agreed that there are only three plane figures which can fill a space, the triangle, the square, and the hexagon, and only two solids, the pyramid [tetrahedron] and the cube."
– Aristotle. *On the Heavens*, volume III

Building blocks by design

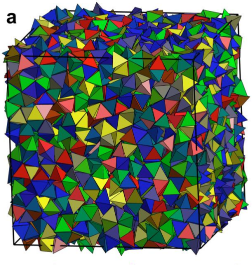


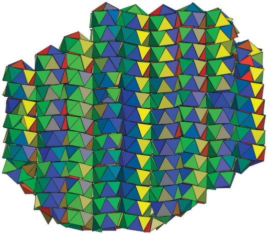
Glotzer and Solomon, Nature Materials 2007

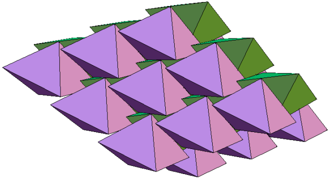
Packing spheres vs. tetrahedra



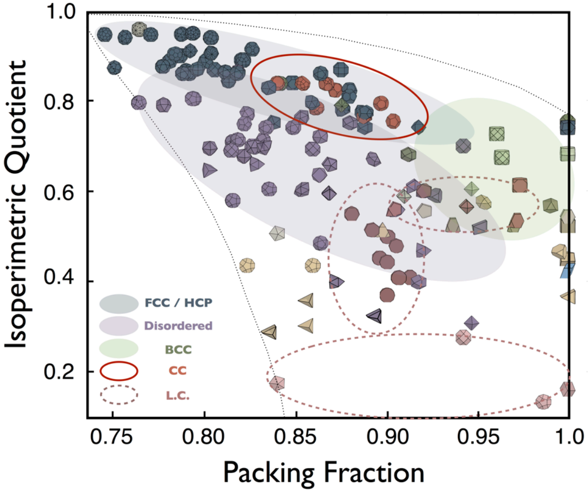
Emergent structure in tetrahedron packing







Packing convex shapes



Ulam's conjecture: balls are worst among convex shapes

Worst packing shapes

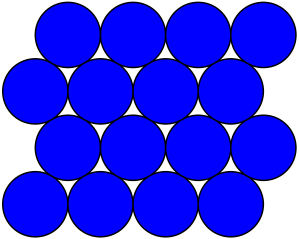
Best packing shapes are trivial

Worst packing shapes

Best packing shapes are trivial

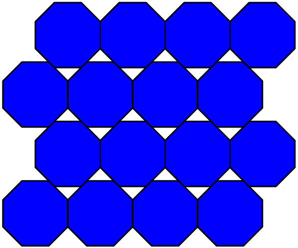
Worst shape is a more interesting question

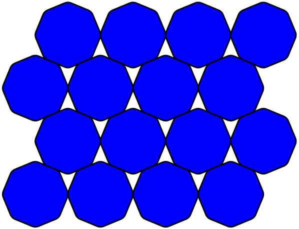
In 2D disks are not worst



$\phi = 0.9069$

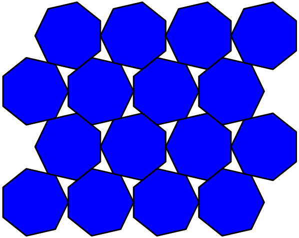
$\phi = 0.9062$



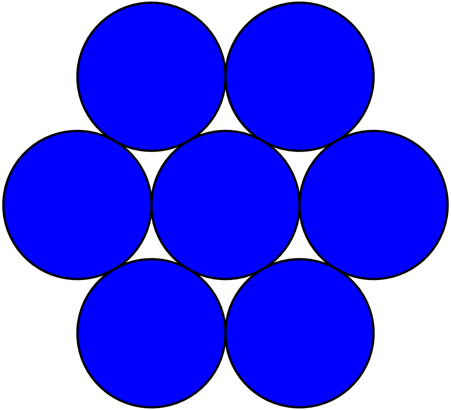


$\phi = 0.9024$

$\phi =$ 0.8926(?)



Why can we improve over circles?



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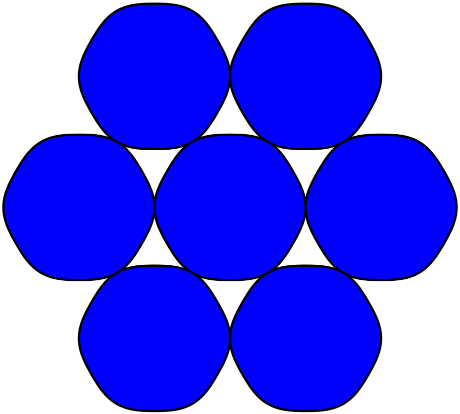

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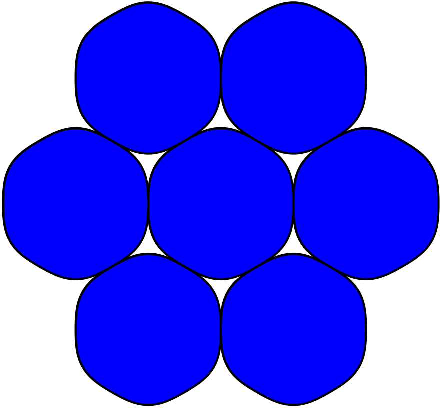

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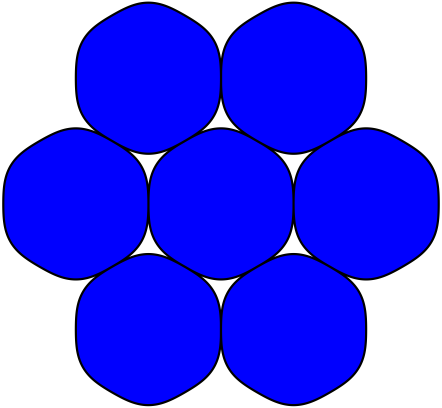


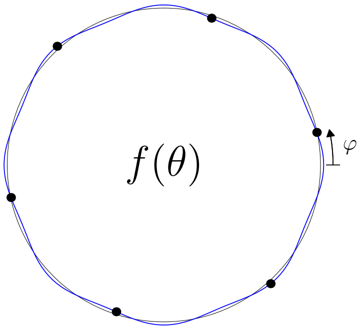
To first order:

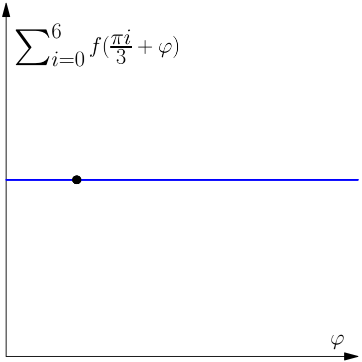
Δ(vol. per particle) $\propto$ avg. deformation in contact dirs.

Δ(vol. of particle) $\propto$ avg. deformation in all dirs.

Can only break even, and make up in higher orders

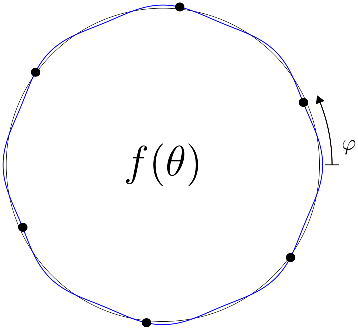
Why can we improve over circles?

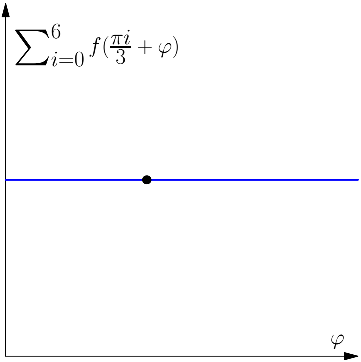




$$f(\theta) = 1 + \epsilon cos(8\theta)$$

Why can we improve over circles?





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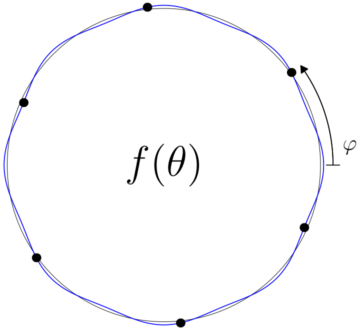

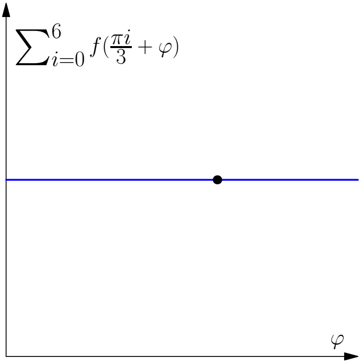

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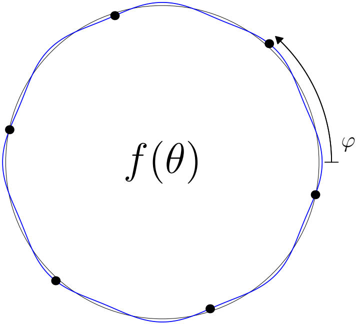

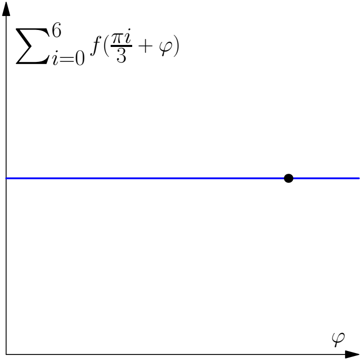

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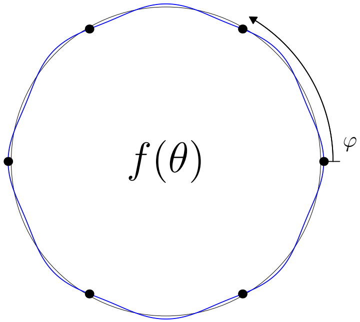
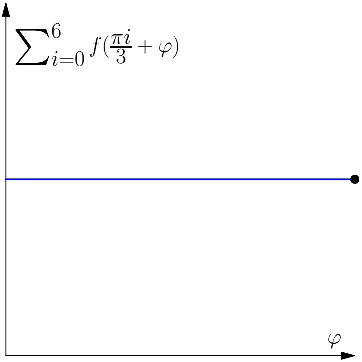
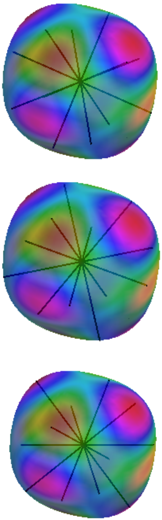




$$f(\theta) = 1 + \epsilon cos(8\theta)$$

Why can we not improve over spheres?



Let $\mathbf{x}_i$, $i = 1, \dots, 12$, be the twelve contact points on the sphere in the f.c.c. packing.

Lemma

Let f be an even function $S^2 \to \mathbb{R}$. $\sum_{i=1}^{12} f(R\mathbf{x}_i)$ is independent of $R \in SO(3)$ if and only if the expansion of $f(\mathbf{x})$ in spherical harmonics terminates at $l = 2$.

K, Adv Math 2014

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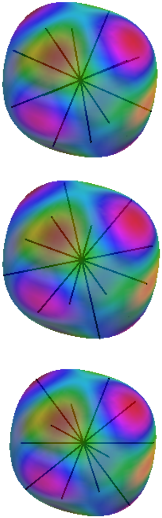

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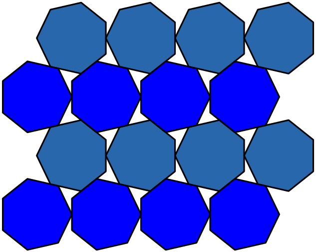
Theorem (K)

The sphere is a local minimum of ϕ, the packing density, among convex, centrally symmetric bodies.

K, Adv Math 2014

Heptagons are locally worst packing (?)



0.8926(?)

Theorem (K)

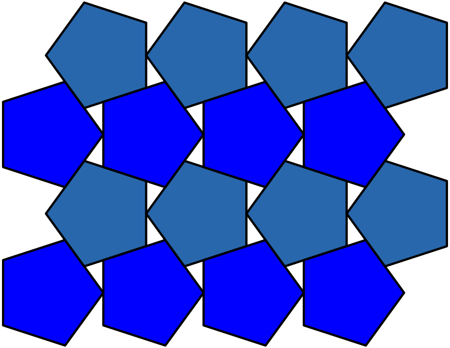
Any convex body sufficiently close to the regular heptagon can be packed at a filling fraction at least that of the "double lattice" packing of regular heptagons.

It is not proven, but highly likely, that the "double lattice" packing is the densest packing of regular heptagons.

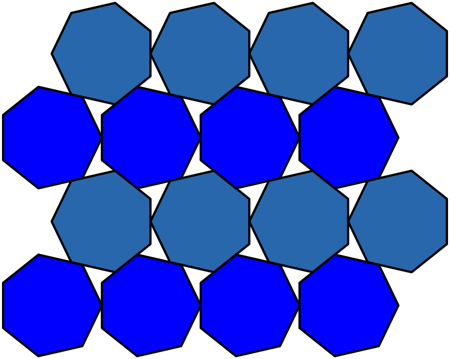
K, Geometry & Topology 2015

Local optimality of the double lattice

Work with Wöden Kusner (TU Graz)



$\phi = 0.9213$



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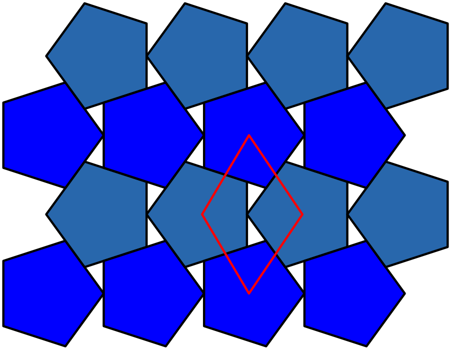
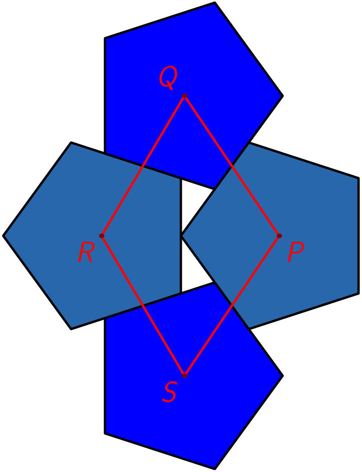
Pentagons

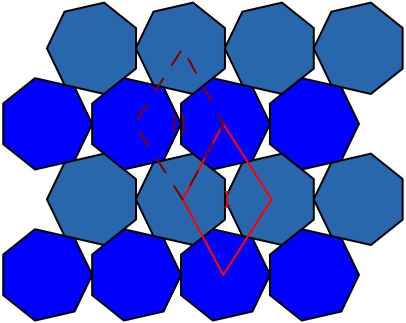


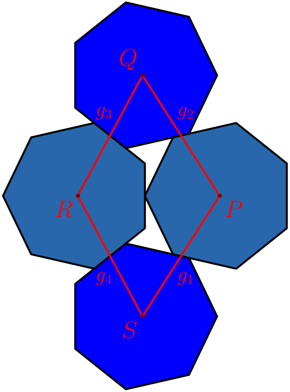


This configuration is a local minimum among nonoverlapping configurations of $\mathrm{area}(SPQR)$.

K and Kusner, Discrete Comput. Geom. 2016

Heptagons

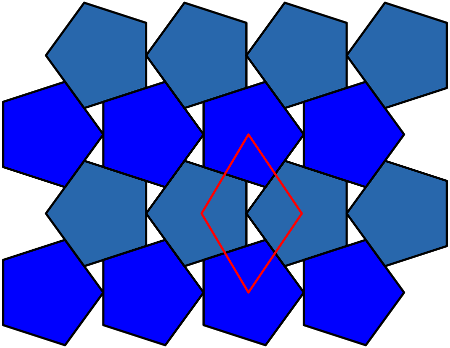




This is not a local minimum of $\mathrm{area}(SPQR)$.
But it is a local minimum of $\mathrm{area}(SPQR) + \sum_{i=1}^{4} g_i$,
where g_i are such that, e.g., $g_3^{(I)} + g_3^{(II)} = 0$.

K and Kusner, Discrete Comput. Geom. 2016

Local optimality of the double lattice



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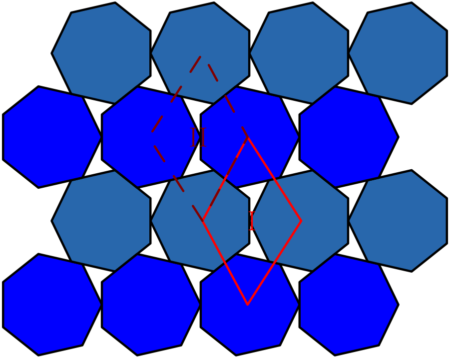

$\phi = 0.8926$

The same method works for (almost) any convex polygon and shows the “double lattice” construction gives locally optimal packings.

K and Kusner, Discrete Comput. Geom. 2016